The Standard Model Lagrangian (including neutrino mass terms)

From "An Introduction to the Standard Model of Particle Physics, 2nd Edition",
W. N. Cottingham and D. A. Greenwood, Cambridge University Press, Cambridge, 2007,
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\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}) - \frac{1}{2}tr(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}) && \text{(U(1), SU(2) and SU(3) gauge terms)}\\
& +(\bar\nu_L,\bar e_L)\,\tilde\sigma^\mu iD_\mu \begin{pmatrix}\nu_L\\ e_L\end{pmatrix} + \bar e_R\sigma^\mu iD_\mu e_R + \bar\nu_R\sigma^\mu iD_\mu\nu_R && \text{(lepton dynamical term)}\\
& -\frac{\sqrt2}{v}\left[(\bar\nu_L,\bar e_L)\,\phi M^e e_R + \bar e_R M^{e*}\bar\phi\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\right] && \text{(electron, muon, tauon mass term)}\\
& -\frac{\sqrt2}{v}\left[(-\bar e_L,\bar\nu_L)\,\phi^* M^\nu \nu_R + \bar\nu_R M^{\nu*}\phi^T\begin{pmatrix}-e_L\\ \nu_L\end{pmatrix}\right] && \text{(neutrino mass term)}\\
& +(\bar u_L,\bar d_L)\,\tilde\sigma^\mu iD_\mu \begin{pmatrix}u_L\\ d_L\end{pmatrix} + \bar u_R\sigma^\mu iD_\mu u_R + \bar d_R\sigma^\mu iD_\mu d_R && \text{(quark dynamical term)}\\
& -\frac{\sqrt2}{v}\left[(\bar u_L,\bar d_L)\,\phi M^d d_R + \bar d_R M^{d*}\bar\phi\begin{pmatrix}u_L\\ d_L\end{pmatrix}\right] && \text{(down, strange, bottom mass term)}\\
& -\frac{\sqrt2}{v}\left[(-\bar d_L,\bar u_L)\,\phi^* M^u u_R + \bar u_R M^{u*}\phi^T\begin{pmatrix}-d_L\\ u_L\end{pmatrix}\right] && \text{(up, charmed, top mass term)}\\
& +\overline{(D_\mu\phi)}D^\mu\phi - m_h^2[\bar\phi\phi - v^2/2]^2/v^2, && \text{(Higgs dynamical and mass term)}\\
& +\text{(Hermitian conjugate of some terms).} && (1)
\end{aligned}
$$

where $\bar\psi = \psi^\dagger$, and the derivative operators are

$$D_\mu\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} = \left[\partial_\mu - \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu\right]\begin{pmatrix}\nu_L\\ e_L\end{pmatrix},\quad D_\mu\nu_R = \partial_\mu\nu_R,\quad D_\mu e_R = [\partial_\mu - ig_1B_\mu]\,e_R, \tag{2}$$

$$D_\mu\begin{pmatrix}u_L\\ d_L\end{pmatrix} = \left[\partial_\mu + \frac{ig_1}{6}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu + ig\mathbf{G}_\mu\right]\begin{pmatrix}u_L\\ d_L\end{pmatrix}, \tag{3}$$

$$D_\mu u_R = \left[\partial_\mu + \frac{i2g_1}{3}B_\mu + ig\mathbf{G}_\mu\right]u_R,\quad D_\mu d_R = \left[\partial_\mu - \frac{ig_1}{3}B_\mu + ig\mathbf{G}_\mu\right]d_R, \tag{4}$$

$$D_\mu\phi = \left[\partial_\mu + \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu\right]\phi. \tag{5}$$

ϕ is a 2-component complex Higgs field. Since $\mathcal{L}$ is $SU(2)$ gauge invariant, a gauge can be chosen so ϕ has the form

$$\phi^T = (0, v+h)/\sqrt2\,,\qquad <\phi>_0^T = \text{(expectation value of } \phi) = (0,v)/\sqrt2\,, \tag{6}$$

where v is a real constant such that $\mathcal{L}_\phi = \overline{(\partial_\mu\phi)}\partial^\mu\phi - m_h^2[\bar\phi\phi - v^2/2]^2/v^2$ is minimized, and h is a residual Higgs field. B_μ, $\mathbf{W}_\mu$ and $\mathbf{G}_\mu$ are the gauge boson vector potentials, and $\mathbf{W}_\mu$ and $\mathbf{G}_\mu$ are composed of 2×2 and 3×3 traceless Hermitian matrices. Their associated field tensors are

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,\quad \mathbf{W}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu + ig_2(\mathbf{W}_\mu\mathbf{W}_\nu - \mathbf{W}_\nu\mathbf{W}_\mu)/2,\quad \mathbf{G}_{\mu\nu} = \partial_\mu\mathbf{G}_\nu - \partial_\nu\mathbf{G}_\mu + ig(\mathbf{G}_\mu\mathbf{G}_\nu - \mathbf{G}_\nu\mathbf{G}_\mu). \tag{7}$$

The non-matrix $A_\mu, Z_\mu, W_\mu^\pm$ bosons are mixtures of $\mathbf{W}_\mu$ and B_μ components, according to the weak mixing angle θ_w,

$$A_\mu = W_{11\mu}sin\theta_w + B_\mu cos\theta_w,\qquad Z_\mu = W_{11\mu}cos\theta_w - B_\mu sin\theta_w,\qquad W_\mu^+ = W_\mu^{-*} = W_{12\mu}/\sqrt2, \tag{8}$$

$$B_\mu = A_\mu cos\theta_w - Z_\mu sin\theta_w,\quad W_{11\mu} = -W_{22\mu} = A_\mu sin\theta_w + Z_\mu cos\theta_w,\quad W_{12\mu} = W_{21\mu}^* = \sqrt2\,W_\mu^+,\quad sin^2\theta_w = .2315(4). \tag{9}$$

The fermions include the leptons e_R, e_L, ν_R, ν_L and quarks u_R, u_L, d_R, d_L. They all have implicit 3-component generation indices, $e_i=(e,\mu,\tau)$, $\nu_i=(\nu_e,\nu_\mu,\nu_\tau)$, $u_i=(u,c,t)$, $d_i=(d,s,b)$, which contract into the fermion mass matrices $M_{ij}^e, M_{ij}^\nu, M_{ij}^u, M_{ij}^d$, and implicit 2-component indices which contract into the Pauli matrices,

$$\sigma^\mu = \left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right],\quad \tilde\sigma^\mu = [\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3],\quad tr(\sigma^i) = 0,\quad \sigma^{\mu\dagger} = \sigma^\mu,\quad tr(\sigma^\mu\sigma^\nu) = 2\delta^{\mu\nu}. \tag{10}$$

The quarks also have implicit 3-component color indices which contract into $\mathbf{G}_\mu$. So $\mathcal{L}$ really has implicit sums over 3-component generation indices, 2-component Pauli indices, 3-component color indices in the quark terms, and 2-component $SU(2)$ indices in $(\bar\nu_L,\bar e_L)$, $(\bar u_L,\bar d_L)$, $(-\bar e_L,\bar\nu_L)$, $(-\bar d_L,\bar u_L)$, $\bar\phi$, $\mathbf{W}_\mu$, $\binom{\nu_L}{e_L}$, $\binom{u_L}{d_L}$, $\binom{-e_L}{\nu_L}$, $\binom{-d_L}{u_L}$, ϕ.

The electroweak and strong coupling constants, Higgs vacuum expectation value (VEV), and Higgs mass are,

$$g_1 = e/cos\theta_w, \quad g_2 = e/sin\theta_w, \quad g = 3.892e, \quad v = \sqrt{2}\cdot 180\,GeV = 254\,GeV, \quad m_h \sim 115 - 180 GeV? \tag{11}$$

where $e\!=\!\sqrt{4\pi\alpha\hbar c}\!=\!\sqrt{4\pi/137}$ in natural units. Using (5,6) and rewriting some things gives the mass of $A_\mu, Z_\mu, W^\pm_\mu$,

$$-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}tr(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}) = -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}\mathcal{W}^-_{\mu\nu}\mathcal{W}^{+\mu\nu} + \begin{pmatrix}\text{higher}\\ \text{order terms}\end{pmatrix}, \tag{12}$$

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu, \quad \mathcal{W}^\pm_{\mu\nu} = D_\mu W^\pm_\nu - D_\nu W^\pm_\mu, \quad D_\mu W^\pm_\nu = [\partial_\mu \pm ieA_\mu]W^\pm_\nu, \tag{13}$$

$$D_\mu <\phi>_0 = \frac{iv}{\sqrt{2}}\begin{pmatrix} g_2 W_{12\mu}/2 \\ g_1 B_\mu/2 + g_2 W_{22\mu}/2 \end{pmatrix} = \frac{ig_2 v}{2}\begin{pmatrix} W_{12\mu}/\sqrt{2} \\ (B_\mu sin\theta_w/cos\theta_w + W_{22\mu})/\sqrt{2} \end{pmatrix} = \frac{ig_2 v}{2}\begin{pmatrix} W^+_\mu \\ -Z_\mu/\sqrt{2}\,cos\theta_w \end{pmatrix}, \tag{14}$$

$$\Rightarrow \quad m_A = 0, \quad m_{W^\pm} = g_2 v/2 = 80.425(38)GeV, \quad m_Z = g_2 v/2cos\theta_w = 91.1876(21)GeV. \tag{15}$$

Ordinary 4-component Dirac fermions are composed of the left and right handed 2-component fields,

$$e = \begin{pmatrix} e_{L1} \\ e_{R1} \end{pmatrix}, \ \nu_e = \begin{pmatrix} \nu_{L1} \\ \nu_{R1} \end{pmatrix}, \ u = \begin{pmatrix} u_{L1} \\ u_{R1} \end{pmatrix}, \ d = \begin{pmatrix} d_{L1} \\ d_{R1} \end{pmatrix}, \quad \text{(electron, electron neutrino, up and down quark)} \tag{16}$$

$$\mu = \begin{pmatrix} e_{L2} \\ e_{R2} \end{pmatrix}, \ \nu_\mu = \begin{pmatrix} \nu_{L2} \\ \nu_{R2} \end{pmatrix}, \ c = \begin{pmatrix} u_{L2} \\ u_{R2} \end{pmatrix}, \ s = \begin{pmatrix} d_{L2} \\ d_{R2} \end{pmatrix}, \quad \text{(muon, muon neutrino, charmed and strange quark)} \tag{17}$$

$$\tau = \begin{pmatrix} e_{L3} \\ e_{R3} \end{pmatrix}, \ \nu_\tau = \begin{pmatrix} \nu_{L3} \\ \nu_{R3} \end{pmatrix}, \ t = \begin{pmatrix} u_{L3} \\ u_{R3} \end{pmatrix}, \ b = \begin{pmatrix} d_{L3} \\ d_{R3} \end{pmatrix}, \quad \text{(tauon, tauon neutrino, top and bottom quark)} \tag{18}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix} \qquad \text{where} \ \ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2Ig^{\mu\nu}. \quad \text{(Dirac gamma matrices in chiral representation)} \tag{19}$$

The corresponding antiparticles are related to the particles according to $\psi^c = -i\gamma^2\psi^*$ or $\psi^c_L = -i\sigma^2\psi^*_R$, $\psi^c_R = i\sigma^2\psi^*_L$. The fermion charges are the coefficients of A_μ when (9,11) are substituted into either the left or right handed derivative operators (2-4). The fermion masses are the singular values of the 3×3 fermion mass matrices M^ν, M^e, M^u, M^d,

$$M^e = \mathbf{U}^{e\dagger}_L\begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}\mathbf{U}^e_R, \ \ M^\nu = \mathbf{U}^{\nu\dagger}_L\begin{pmatrix} m_{\nu_e} & 0 & 0 \\ 0 & m_{\nu_\mu} & 0 \\ 0 & 0 & m_{\nu_\tau} \end{pmatrix}\mathbf{U}^\nu_R, \ \ M^u = \mathbf{U}^{u\dagger}_L\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}\mathbf{U}^u_R, \ \ M^d = \mathbf{U}^{d\dagger}_L\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}\mathbf{U}^d_R, \tag{20}$$

$$m_e = .510998910(13)MeV, \quad m_{\nu_e} \sim .001 - 2eV, \quad m_u = 1.5 - 3.3MeV, \quad m_d = 3.5 - 6MeV, \tag{21}$$

$$m_\mu = 105.658367(4)MeV, \quad m_{\nu_\mu} \sim .001 - 2eV, \quad m_c = 1.16 - 1.34GeV, \quad m_s = 70 - 130MeV, \tag{22}$$

$$m_\tau = 1776.84(17)MeV, \quad m_{\nu_\tau} \sim .001 - 2eV, \quad m_t = 169 - 174GeV, \quad m_b = 4.13 - 4.37GeV, \tag{23}$$

where the $\mathbf{U}$s are 3×3 unitary matrices ($\mathbf{U}^{-1} = \mathbf{U}^\dagger$). Consequently the "true fermions" with definite masses are actually linear combinations of those in $\mathcal{L}$, or conversely the fermions in $\mathcal{L}$ are linear combinations of the true fermions,

$$e'_L = \mathbf{U}^e_L e_L, \quad e'_R = \mathbf{U}^e_R e_R, \quad \nu'_L = \mathbf{U}^\nu_L \nu_L, \quad \nu'_R = \mathbf{U}^\nu_R \nu_R, \quad u'_L = \mathbf{U}^u_L u_L, \quad u'_R = \mathbf{U}^u_R u_R, \quad d'_L = \mathbf{U}^d_L d_L, \quad d'_R = \mathbf{U}^d_R d_R, \tag{24}$$

$$e_L = \mathbf{U}^{e\dagger}_L e'_L, \quad e_R = \mathbf{U}^{e\dagger}_R e'_R, \quad \nu_L = \mathbf{U}^{\nu\dagger}_L \nu'_L, \quad \nu_R = \mathbf{U}^{\nu\dagger}_R \nu'_R, \quad u_L = \mathbf{U}^{u\dagger}_L u'_L, \quad u_R = \mathbf{U}^{u\dagger}_R u'_R, \quad d_L = \mathbf{U}^{d\dagger}_L d'_L, \quad d_R = \mathbf{U}^{d\dagger}_R d'_R. \tag{25}$$

When $\mathcal{L}$ is written in terms of the true fermions, the $\mathbf{U}$s fall out except in $\bar u'_L \mathbf{U}^u_L \tilde\sigma^\mu W^\pm_\mu \mathbf{U}^{d\dagger}_L d'_L$ and $\bar\nu'_L \mathbf{U}^\nu_L \tilde\sigma^\mu W^\pm_\mu \mathbf{U}^{e\dagger}_L e'_L$. Because of this, and some absorption of constants into the fermion fields, the parameters in the $\mathbf{U}$s are entirely contained in only four components of the Cabibbo-Kobayashi-Maskawa matrix $\mathbf{V}^q = \mathbf{U}^u_L \mathbf{U}^{d\dagger}_L$ and four components of $\mathbf{V}^l = \mathbf{U}^\nu_L \mathbf{U}^{e\dagger}_L$. The unitary matrices $\mathbf{V}^q$ and $\mathbf{V}^l$ are often parameterized as

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}\begin{pmatrix} e^{-i\delta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta/2} \end{pmatrix}\begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}\begin{pmatrix} e^{i\delta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta/2} \end{pmatrix}\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & \\ 0 & 0 & 1 \end{pmatrix}, \quad c_j = \sqrt{1 - s_j^2}, \tag{26}$$

$$\delta^q = 57(14)\,\text{deg}, \quad s^q_{12} = 0.2243(16), \quad s^q_{23} = 0.0413(15), \quad s^q_{13} = 0.0037(5), \tag{27}$$

$$\delta^l = ?, \quad s^l_{12} = 0.57(3), \quad s^l_{23} = 0.7(1), \quad s^l_{13} = 0.0(2). \tag{28}$$

$\mathcal{L}$ is invariant under a $U(1)\otimes SU(2)$ gauge transformation with $U^{-1} = U^\dagger$, $detU = 1$, θ real,

$$\mathbf{W}_\mu \to U\mathbf{W}_\mu U^\dagger - (2i/g_2)U\partial_\mu U^\dagger, \quad \mathbf{W}_{\mu\nu} \to U\mathbf{W}_{\mu\nu}U^\dagger, \quad B_\mu \to B_\mu + (2/g_1)\partial_\mu\theta, \quad B_{\mu\nu} \to B_{\mu\nu}, \quad \phi \to e^{-i\theta}U\phi, \tag{29}$$

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \to e^{i\theta}U\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad \begin{pmatrix} u_L \\ d_L \end{pmatrix} \to e^{-i\theta/3}U\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{matrix} \nu_R \to \nu_R, & u_R \to e^{-4i\theta/3}u_R, \\ e_R \to e^{2i\theta}e_R, & d_R \to e^{2i\theta/3}d_R, \end{matrix} \tag{30}$$

and under an $SU(3)$ gauge transformation with $V^{-1} = V^\dagger$, $detV = 1$,

$$\mathbf{G}_\mu \to V\mathbf{G}_\mu V^\dagger - (i/g)V\partial_\mu V^\dagger, \quad \mathbf{G}_{\mu\nu} \to V\mathbf{G}_{\mu\nu}V^\dagger, \quad u_L \to Vu_L, \quad d_L \to Vd_L, \quad u_R \to Vu_R, \quad d_R \to Vd_R. \tag{31}$$