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EXTENSIONS OF THE EINSTEIN-SCHRÖDINGER NON-SYMMETRIC THEORY OF GRAVITY

by

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Contents

Abstract

We modify the Einstein-Schrödinger theory to include a cosmological constant Λ_z which multiplies the symmetric metric. The cosmological constant Λ_z is assumed to be nearly cancelled by Schrödinger's cosmological constant Λ_b which multiplies the nonsymmetric fundamental tensor, such that the total $\Lambda = \Lambda_z + \Lambda_b$ matches measurement. The resulting theory becomes exactly Einstein-Maxwell theory in the limit as $|\Lambda_z| \to \infty$. For $|\Lambda_z| \sim 1/(\text{Planck length})^2$ the field equations match the ordinary Einstein and Maxwell equations except for extra terms which are $< 10^{-16}$ of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. Additional fields can be included in the Lagrangian, and these fields may couple to the symmetric metric and the electromagnetic vector potential, just as in Einstein-Maxwell theory. The ordinary Lorentz force equation is obtained by taking the divergence of the Einstein equations when sources are included. The Einstein-Infeld-Hoffmann (EIH) equations of motion match the equations of motion for Einstein-Maxwell theory to Newtonian/Coulombian order, which proves the existence of a Lorentz force without requiring sources. An exact charged solution matches the Reissner-Nordström solution except for additional terms which are $\sim 10^{-66}$ of the usual terms for worst-case radii accessible to measurement. An exact electromagnetic plane-wave solution is identical to its counterpart in Einstein-Maxwell theory. Pericenter advance, deflection of light and time delay of light have a fractional difference of $<10^{-56}$ compared to Einstein-Maxwell theory for worst-case parameters. When a spin-1/2 field is included in the Lagrangian, the theory gives the ordinary Dirac equation, and the charged solution results in fractional shifts of $<10^{-50}$ in Hydrogen atom energy levels. Newman-Penrose methods are used to derive an exact solution of the connection equations, and to show that the charged solution is Petrov type-D like the Reissner-Nordström solution. The Newman-Penrose asymptotically flat $\mathcal{O}(1/r^2)$ expansion of the field equations is shown to match Einstein-Maxwell theory. Finally we generalize the theory to non-Abelian fields, and show that a special case of the resulting theory closely approximates Einstein-Weinberg-Salam theory.

Chapter 1

Introduction

Einstein-Maxwell theory is the standard theory which couples general relativity with electrodynamics. In this theory, space-time geometry and gravity are described by a metric $g_{\mu\nu}$ which is symmetric ($g_{\mu\nu} = g_{\nu\mu}$), and the electromagnetic field $F_{\mu\nu}$ is antisymmetric ($F_{\mu\nu} = -F_{\nu\mu}$). The fact that these two fields could be combined together into one second rank tensor was noticed long ago by researchers looking for a more unified description of the physical laws. The Einstein-Schrödinger theory is a generalization of vacuum general relativity which allows a nonsymmetric field $N_{\mu\nu}$ in place of the symmetric $g_{\mu\nu}$. The theory without a cosmological constant was first proposed by Einstein and Straus[1, 2, 3, 4, 5]. Schrödinger later showed that it could be derived from a very simple Lagrangian density if a cosmological constant Λ_b was included[6, 7, 8]. Einstein and Schrödinger suspected that the theory might include electrodynamics, where the nonsymmetric "fundamental tensor" $N_{\mu\nu}$ contained both the metric and electromagnetic field. However, this hope was dashed when it was found that the theory did not predict a Lorentz force between charged particles[9, 10].

In this dissertation we describe a simple modification of the Einstein-Schrödinger theory[11, 12, 13, 14] which closely approximates Einstein-Maxwell theory, and where the Lorentz force does occur. The modification involves the addition of a second cosmological term $\Lambda_z g_{\mu\nu}$ to the field equations, where $g_{\mu\nu}$ is the symmetric metric. We assume this term is nearly canceled by Schrödinger's "bare" cosmological term $\Lambda_b N_{\mu\nu}$, where $N_{\mu\nu}$ is the nonsymmetric fundamental tensor. The total "physical" cosmological constant $\Lambda = \Lambda_b + \Lambda_z$ can then be made to match cosmological measurements of the accelerating universe.

The origin of our Λ_z is unknown. One possibility is that Λ_z could arise from vacuum fluctuations, an idea discussed by many authors[15, 16, 17, 18]. Zero-point fluctuations are essential to both quantum electrodynamics and the Standard Model, and are thought to be the cause of the Casimir force[16] and other effects. With this interpretation, the fine tuning of cosmological constants is not so objectionable because it resembles mass/charge/field-strength renormalization in quantum electrodynamics. For example, to cancel electron self-energy in quantum electrodynamics, the "bare" electron mass becomes large for a cutoff frequency $\omega_c \sim 1/(\text{Planck length})$, and infinite if $\omega_c \to \infty$, but the total "physical" mass remains small. In a similar manner, to cancel zero-point energy in our theory, the "bare" cosmological constant $\Lambda_b \sim \omega_c^4 \times (\text{Planck length})^2$ becomes large if $\omega_c \sim 1/(\text{Planck length})$, and infinite if $\omega_c \to \infty$, but the total "physical" Λ remains small. There are other possible origins of Λ_z. For example Λ_z could arise dynamically, related to the minimum of a potential of some additional field in the theory. Apart from the discussion above, speculation about the origin of Λ_z is outside the scope of this dissertation. Our main goal is to show that the theory closely approximates Einstein-Maxwell theory, and for non-Abelian fields the Einstein-Weinberg-Salam theory (general relativity coupled to electro-weak theory).

Like Einstein-Maxwell theory, our theory can be coupled to additional fields using a symmetric metric $g_{\mu\nu}$ and vector potential A_μ, and it is invariant under a $U(1)$ gauge transformation. The theory does not enlarge the invariance group. When coupled to the Standard Model, the combined Lagrangian is invariant under the usual $U(1)\otimes SU(2)\otimes SU(3)$ gauge group. The usual $U(1)$ gauge term $F^{\mu\nu}F_{\mu\nu}$ is incorporated together with the geometry, and is not explicitly in the Lagrangian. The non-Abelian version of the theory can also be coupled to the Standard Model, in which case both the $U(1)$ and $SU(2)$ gauge terms are incorporated together with the geometry. This is done much as it is done in [19, 20] with Bonnor's theory. Whether the $SU(3)$ gauge term of the Standard model could also be incorporated with a larger gauge group, or by using higher space-time dimensions, is beyond the scope of this dissertation.

The Abelian version of our theory is similar to [21, 22] but with the opposite sign of Λ_b and Λ_z. Because of this difference our theory involves Hermitian fields instead of real fields, and the spherically symmetric solutions have much different properties near the origin and do not come in an infinite set. The Abelian version of our theory is also roughly the electromagnetic dual of another theory[23, 24, 25, 26]. Compared to all of these other theories, our theory also allows coupling to additional fields (sources), and it allows $\Lambda\neq 0$, and it is derived from a Lagrangian density which incorporates a new type of non-symmetric Ricci tensor with different invariance properties.

Many other modifications of the Einstein-Schrödinger theory have been considered. For example in Bonnor's theory[27, 28] the antisymmetric part of the fundamental tensor $N_{[\tau\rho]}$ or its dual is taken to be the electromagnetic field, and a Lorentz force is derived, but only because a $\sqrt{-N}N^{\dashv[\rho\tau]}N_{[\tau\rho]}$ term is appended onto the usual Lagrangian density. Other theories include an assortment of additional terms in the Lagrangian density[29, 30]. Such theories lack the mathematical simplicity of the original Einstein-Schrödinger theory, and for that reason they seem unsatisfying. This criticism seems less applicable to our theory because there are such good motivations for including a $\Lambda_z\sqrt{-g}$ term in the Lagrangian density.

Some previous work[31, 32, 33] shows that the original Einstein-Schrödinger theory has problems with negative energy "ghosts". As will be seen in §2.4, this problem is avoided in our theory in an unusual way. In [31, 32, 33] referenced above, the electromagnetic field is assumed to be an independent field added onto the Lagrangian, and it is unrelated to $N_{[\nu\mu]}$. Because of the coupling of $N_{\nu\mu}$ to the electromagnetic field in such theories, there would be observable violations[34, 35, 36] of the principle of equivalence for values of $N_{[\nu\mu]}$ which occur in the theory. Such problems do not apply in our theory, mainly because we assume a symmetric metric which is defined in terms of $N_{\nu\mu}$, and it is this symmetric metric which appears in Maxwell's equations, and any coupling to additional fields. Such problems are also avoided in our theory partly because of the small values of $N_{[\nu\mu]}$ which occur.

In most previous work on the original Einstein-Schrödinger theory, the electromagnetic field is assumed to be the dual of $N_{[\tau\rho]}$. Even though this is the same definition used in [9, 10] to show there is no Lorentz force, several authors claim that a Lorentz-like force can be demonstrated[37, 38, 39]. However, the solutions[40, 41, 42] that must be used for test particles have bad asymptotic behavior, such as a radial electric field which is independent of radius at large distances. Our theory uses a different definition of the electromagnetic field, and it has satisfactory exact solutions for both an electric monopole as in §3.1, and an electromagnetic plane-wave as in §3.2.

Many others have contributed to the Einstein-Schrödinger theory. Of particular significance to our modified theory are contributions related to the choice of metric[43, 44, 45, 37], the generalized contracted Bianchi identity[43, 44, 45], the inclusion of sources[45, 19, 37], and exact solutions with a cosmological constant[46, 47].

This dissertation is organized as follows. In §2.1 we discuss the Lagrangian density. In §2.2-§2.4 we derive the field equations and quantify how closely they approximate the field equations of Einstein-Maxwell theory. In §3.1 we present an exact charged solution and show that it closely approximates the Reissner-Nordström solution. In §3.2 we present an exact electromagnetic plane-wave solution which is identical to its counterpart in Einstein-Maxwell theory. In §4.1 we derive the ordinary Lorentz force equation by taking the divergence of the Einstein equations when sources are included. In §4.2 we use the Lorentz force equation to derive the equations of motion for charged and neutral particles around the charged solution. In §4.3 we derive the Lorentz force using the EIH method, which requires no sources in the Lagrangian. In §5.1-§5.2 we calculate pericenter advance, deflection of light, and time delay of light, and compare the results to Einstein-Maxwell theory. In §5.3 we include a spin-1/2 field in the Lagrangian and estimate the shift in Hydrogen atom energy levels for this theory as compared with Einstein-Maxwell theory. In §6.1 we represent the exact field equations in Newman-Penrose tetrad form, and use this to derive an exact solution of the connection equations, and to show that the charged solution is Petrov type-D like the Reissner-Nordström solution. In §6.2 we derive the Newman-Penrose asymptotically flat $\mathcal{O}(1/r^2)$ expansion of the field equations, and compare the results to Einstein-Maxwell theory. In §7.1-§7.3 we consider a generalization of our theory to non-Abelian fields, and show that a special case of the theory closely approximates Einstein-Weinberg-Salam theory.

Chapter 2

Extension of the Einstein-Schrödinger theory for Abelian fields

2.1 The Lagrangian density

Einstein-Maxwell theory can be derived from a Palatini Lagrangian density, meaning that it depends on a connection $\Gamma^\lambda_{\rho\tau}$ as well as the metric $g_{\rho\tau}$,

$$\begin{aligned}\mathcal{L}(\Gamma^\lambda_{\rho\tau}, g_{\rho\tau}, A_\nu) &= -\frac{1}{16\pi}\sqrt{-g}\,[\,g^{\mu\nu}R_{\nu\mu}(\Gamma) + 2\Lambda_b\,]\\ &\quad +\frac{1}{4\pi}\sqrt{-g}A_{[\alpha,\rho]}g^{\alpha\mu}g^{\rho\nu}A_{[\mu,\nu]} + \mathcal{L}_m(u^\nu, \psi, g_{\mu\nu}, A_\nu\cdots).\end{aligned} \tag{2.1}$$

Here Λ_b is a bare cosmological constant. The $\mathcal{L}_m$ term couples the metric $g_{\mu\nu}$ and electromagnetic potential A_μ to additional fields, such as a hydrodynamic velocity vector u^ν, spin-1/2 wavefunction ψ, or perhaps the other fields of the Standard Model. The original Einstein-Schrödinger theory allows a nonsymmetric $N_{\mu\nu}$ and $\widehat{\Gamma}^\lambda_{\rho\tau}$ in place of the symmetric $g_{\mu\nu}$ and $\Gamma^\lambda_{\rho\tau}$, and excludes the $\sqrt{-g}A_{[\alpha,\rho]}g^{\alpha\mu}g^{\rho\nu}A_{[\mu,\nu]}$ term (see Appendix M). Our "Λ-renormalized" Einstein-Schrödinger (LRES) theory introduces an additional cosmological term $\sqrt{-g}\Lambda_z$,

$$\begin{aligned}\mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau}, N_{\rho\tau}) \;&=\; -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) + (n-2)\Lambda_b\right]\\ &\quad -\frac{1}{16\pi}\sqrt{-g}\,(n-2)\Lambda_z + \mathcal{L}_m(u^\nu,\psi,g_{\mu\nu},A_\nu\ldots),\end{aligned} \tag{2.2}$$

where $\Lambda_b \approx -\Lambda_z$ so that the total Λ matches astronomical measurements[48],

$$\Lambda = \Lambda_b + \Lambda_z \approx \;\; 10^{-56}\mathrm{cm}^{-2}, \tag{2.3}$$

and the physical metric and electromagnetic potential are defined to be

$$\sqrt{-g}\,g^{\mu\nu} = \sqrt{-N}N^{\dashv(\mu\nu)}, \qquad\qquad A_\nu = \widehat{\Gamma}^\sigma_{[\nu\sigma]}/[(n-1)\sqrt{-2\Lambda_b}\,]. \tag{2.4}$$

Equation (2.4) defines $g^{\mu\nu}$ unambiguously because $\sqrt{-g} = [-det(\sqrt{-g}\,g^{\mu\nu})]^{1/(n-2)}$. Here and throughout this paper we use geometrized units with $c = G = 1$, the symbols ( ) and [ ] around indices indicate symmetrization and antisymmetrization, $g = det(g_{\mu\nu})$, $N = det(N_{\mu\nu})$, and $N^{\dashv\sigma\nu}$ is the inverse of $N_{\nu\mu}$ such that $N^{\dashv\sigma\nu}N_{\nu\mu} = \delta^\sigma_\mu$. The dimension is assumed to be n=4, but "n" is retained in the equations to show how easily the theory can be generalized. The $\mathcal{L}_m$ term is not to include a $\sqrt{-g}A_{[\alpha,\beta]}g^{\alpha\mu}g^{\beta\nu}A_{[\mu,\nu]}$ part but may contain the rest of the Standard Model. In (2.2), $\mathcal{R}_{\nu\mu}(\widehat{\Gamma})$ is a form of non-symmetric Ricci tensor with special invariance properties to be discussed later,

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) = \widehat{\Gamma}^\alpha_{\nu\mu,\alpha} - \widehat{\Gamma}^\alpha_{(\alpha(\nu),\mu)} + \widehat{\Gamma}^\sigma_{\nu\mu}\widehat{\Gamma}^\alpha_{(\alpha\sigma)} - \widehat{\Gamma}^\sigma_{\nu\alpha}\widehat{\Gamma}^\alpha_{\sigma\mu} - \widehat{\Gamma}^\tau_{[\tau\nu]}\widehat{\Gamma}^\alpha_{[\alpha\mu]}/(n-1). \tag{2.5}$$

This tensor reduces to the ordinary Ricci tensor when $\widehat{\Gamma}^\alpha_{\nu\mu}$ is the Christoffel connection with $\widehat{\Gamma}^\alpha_{[\nu\mu]} = 0$ and $\widehat{\Gamma}^\alpha_{\alpha[\nu,\mu]} = 0$, as occurs in ordinary general relativity.

It is helpful to decompose $\widehat{\Gamma}^\alpha_{\nu\mu}$ into a new connection $\tilde{\Gamma}^\alpha_{\nu\mu}$, and A_σ from (2.4),

$$\widehat{\Gamma}^\alpha_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu A_\nu - \delta^\alpha_\nu A_\mu)\sqrt{-2\Lambda_b}, \tag{2.6}$$

$$\text{where} \quad \tilde{\Gamma}^\alpha_{\nu\mu} = \widehat{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu \widehat{\Gamma}^\sigma_{[\sigma\nu]} - \delta^\alpha_\nu \widehat{\Gamma}^\sigma_{[\sigma\mu]})/(n-1). \tag{2.7}$$

By contracting (2.7) on the right and left we see that $\tilde{\Gamma}^\alpha_{\nu\mu}$ has the symmetry

$$\tilde{\Gamma}^\alpha_{\nu\alpha} = \widehat{\Gamma}^\alpha_{(\nu\alpha)} = \tilde{\Gamma}^\alpha_{\alpha\nu}, \tag{2.8}$$

so it has only $n^3 - n$ independent components whereas $\widehat{\Gamma}^\alpha_{\nu\alpha}$ had n^3. Substituting (2.6) into (2.5) as in R.17 gives

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) = \mathcal{R}_{\nu\mu}(\tilde{\Gamma}) + 2A_{[\nu,\mu]}\sqrt{-2\Lambda_b}. \tag{2.9}$$

Using (2.9), the Lagrangian density (2.2) can be written in terms of $\tilde{\Gamma}^\alpha_{\nu\mu}$ and A_σ,

$$\begin{aligned}\mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau}, N_{\rho\tau}) &= -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}(\tilde{\mathcal{R}}_{\nu\mu} + 2A_{[\nu,\mu]}\sqrt{-2\Lambda_b}) + (n-2)\Lambda_b\right] \\ &\quad -\frac{1}{16\pi}\sqrt{-g}\,(n-2)\Lambda_z + \mathcal{L}_m(u^\nu, \psi, g_{\mu\nu}, A_\sigma \dots).\end{aligned} \tag{2.10}$$

Here $\tilde{\mathcal{R}}_{\nu\mu} = \mathcal{R}_{\nu\mu}(\tilde{\Gamma})$, and from (2.8,2.5) we have

$$\tilde{\mathcal{R}}_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu,\alpha} - \tilde{\Gamma}^\alpha_{\alpha(\nu,\mu)} + \tilde{\Gamma}^\sigma_{\nu\mu}\tilde{\Gamma}^\alpha_{\sigma\alpha} - \tilde{\Gamma}^\sigma_{\nu\alpha}\tilde{\Gamma}^\alpha_{\sigma\mu}. \tag{2.11}$$

From (2.6,2.8), $\tilde{\Gamma}^\alpha_{\nu\mu}$ and A_ν fully parameterize $\widehat{\Gamma}^\alpha_{\nu\mu}$ and can be treated as independent variables. It is simpler to calculate the field equations by setting $\delta\mathcal{L}/\delta\tilde{\Gamma}^\alpha_{\nu\mu} = 0$ and $\delta\mathcal{L}/\delta A_\nu = 0$ instead of setting $\delta\mathcal{L}/\delta\widehat{\Gamma}^\alpha_{\nu\mu} = 0$, so we will follow this method.

To do quantitative comparisons of this theory to Einstein-Maxwell theory we will need to use some value for Λ_z. One possibility is that Λ_z results from zero-point fluctuations[15, 16, 17, 18], in which case using (2.3) we get

$$\Lambda_b \approx -\Lambda_z \sim C_z\omega_c^4 l_P^2 \sim 10^{66}\text{cm}^{-2}, \tag{2.12}$$

$$\omega_c = (\text{cutoff frequency}) \sim 1/l_P, \tag{2.13}$$

$$C_z = \frac{1}{2\pi}\left(\begin{matrix}\text{fermion}\\ \text{spin states}\end{matrix} - \begin{matrix}\text{boson}\\ \text{spin states}\end{matrix}\right) \sim \frac{60}{2\pi} \tag{2.14}$$

where $l_P = (\text{Planck length}) = 1.6\times 10^{-33}cm$. We will also consider the limit $\omega_c \to \infty$, $|\Lambda_z| \to \infty$, $\Lambda_b \to \infty$ as in quantum electrodynamics, and we will prove that

$$\lim_{|\Lambda_z|\to\infty}\left(\begin{matrix}\Lambda\text{-renormalized}\\ \text{Einstein-Schrödinger theory}\end{matrix}\right) = \left(\begin{matrix}\text{Einstein-Maxwell}\\ \text{theory}\end{matrix}\right). \tag{2.15}$$

The non-symmetric Ricci tensor (2.5) has the following invariance properties

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}^\alpha_{\rho\tau}) = \mathcal{R}_{\mu\nu}(\widehat{\Gamma}^\alpha_{\tau\rho}), \tag{2.16}$$

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}^\alpha_{\rho\tau} + \delta^\alpha_{[\rho}\varphi_{,\tau]}) = \mathcal{R}_{\nu\mu}(\widehat{\Gamma}^\alpha_{\rho\tau}) \quad \text{for an arbitrary } \varphi(x^\sigma). \tag{2.17}$$

From (2.16,2.17), the Lagrangians (2.2,2.10) are invariant under charge conjugation,

$$Q \to -Q,\ A_\sigma \to -A_\sigma,\ \tilde{\Gamma}^\alpha_{\nu\mu} \to \tilde{\Gamma}^\alpha_{\mu\nu},\ \widehat{\Gamma}^\alpha_{\nu\mu} \to \widehat{\Gamma}^\alpha_{\mu\nu},\ N_{\nu\mu} \to N_{\mu\nu},\ N^{\dashv\nu\mu} \to N^{\dashv\mu\nu}, \tag{2.18}$$

and also under an electromagnetic gauge transformation

$$\psi \to \psi e^{i\phi},\quad A_\alpha \to A_\alpha - \frac{\hbar}{Q}\phi_{,\alpha},\quad \tilde{\Gamma}^\alpha_{\rho\tau} \to \tilde{\Gamma}^\alpha_{\rho\tau},\quad \widehat{\Gamma}^\alpha_{\rho\tau} \to \widehat{\Gamma}^\alpha_{\rho\tau} + \frac{2\hbar}{Q}\delta^\alpha_{[\rho}\phi_{,\tau]}\sqrt{-2\Lambda_b}, \tag{2.19}$$

assuming that $\mathcal{L}_m$ is invariant. With $\Lambda_b > 0$, $\Lambda_z < 0$ as in (2.12) then $\tilde{\Gamma}^\alpha_{\nu\mu}$, $\widehat{\Gamma}^\alpha_{\nu\mu}$, $N_{\nu\mu}$ and $N^{\dashv\nu\mu}$ are all Hermitian, $\tilde{\mathcal{R}}_{\nu\mu}$ and $\mathcal{R}_{\nu\mu}(\widehat{\Gamma})$ are Hermitian from (2.16), and $g_{\nu\mu}$, A_σ and $\mathcal{L}$ are real from (2.4,2.2,2.10).

In this theory the metric $g_{\mu\nu}$ from (2.4) is used for measuring space-time intervals, for calculating geodesics, and for raising and lowering of indices. The covariant derivative ";" is always done using the Christoffel connection formed from $g_{\mu\nu}$,

$$\Gamma^\alpha_{\nu\mu} \;=\; \frac{1}{2}\,g^{\alpha\sigma}(g_{\mu\sigma,\nu}+g_{\sigma\nu,\mu}-g_{\nu\mu,\sigma}). \tag{2.20}$$

We will see that taking the divergence of the Einstein equations using (2.20,2.4) gives the ordinary Lorentz force equation. The electromagnetic field is defined in terms of the vector potential (2.4)

$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}. \tag{2.21}$$

However, we will also define another field $f^{\mu\nu}$

$$\sqrt{-g}\,f^{\mu\nu}=\sqrt{-N}N^{\dashv[\nu\mu]}\Lambda_b^{1/2}/\sqrt{2}\,i. \tag{2.22}$$

Then from (2.4), $g^{\mu\nu}$ and $f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}$ are parts of a total field,

$$(\sqrt{-N}/\sqrt{-g}\,)N^{\dashv\nu\mu}=g^{\mu\nu}+f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{2.23}$$

We will see that the field equations require $f_{\mu\nu}\approx F_{\mu\nu}$ to a very high precision. The definitions (2.4) of $g_{\mu\nu}$ and A_ν in terms of the "fundamental" fields $N_{\rho\tau},\widehat\Gamma^\lambda_{\rho\tau}$ may seem unnatural from an empirical viewpoint. On the other hand, our Lagrangian density (2.2) seems simpler than (2.1) of Einstein-Maxwell theory, it contains fewer fields, and these fields have no symmetry restrictions. However, these are all very subjective considerations. It is much more important that our theory closely matches Einstein-Maxwell theory, and hence measurement.

Note that there are many nonsymmetric generalizations of the Ricci tensor besides our version $\mathcal{R}_{\nu\mu}(\widehat\Gamma)$ from (2.5) and the ordinary Ricci tensor $R_{\nu\mu}(\widehat\Gamma)$. For example, we could form any weighted average of $R_{\nu\mu}(\widehat\Gamma)$, $R_{\mu\nu}(\widehat\Gamma)$, $R_{\nu\mu}(\widehat\Gamma^T)$ and $R_{\mu\nu}(\widehat\Gamma^T)$, and then add any linear combination of the tensors $\widehat{\Gamma}^{\alpha}_{\alpha[\nu,\mu]}$, $\widehat{\Gamma}^{\alpha}_{[\nu|\alpha,|\mu]}$, $\widehat{\Gamma}^{\alpha}_{[\nu\mu]}\widehat{\Gamma}^{\sigma}_{[\sigma\alpha]}$, $\widehat{\Gamma}^{\alpha}_{[\nu\sigma]}\widehat{\Gamma}^{\sigma}_{[\mu\alpha]}$, and $\widehat{\Gamma}^{\alpha}_{[\alpha\nu]}\widehat{\Gamma}^{\sigma}_{[\sigma\mu]}$. All of these generalized Ricci tensors would be linear in $\widehat{\Gamma}^{\alpha}_{\nu\mu,\sigma}$, quadratic in $\widehat{\Gamma}^{\alpha}_{\nu\mu}$, and would reduce to the ordinary Ricci tensor when $\widehat{\Gamma}^{\alpha}_{[\nu\mu]}=0$ and $\widehat{\Gamma}^{\alpha}_{\alpha[\nu,\mu]}=0$, as occurs in ordinary general relativity. Even if we limit the tensor to only four terms, there are still eight possibilities. We assert that invariance properties like (2.16,2.17) are the most sensible way to choose among the different alternatives, not criteria such as the number of terms in the expression.

Finally, let us discuss some notation issues. We use the symbol $\Gamma^{\alpha}_{\nu\mu}$ for the Christoffel connection (2.20) whereas Einstein and Schrödinger used it for our $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ and $\widehat{\Gamma}^{\alpha}_{\nu\mu}$ respectively. We use the symbol $g_{\mu\nu}$ for the symmetric metric (2.4) whereas Einstein and Schrödinger used it for our $N_{\mu\nu}$, the nonsymmetric fundamental tensor. Also, to represent the inverse of $N_{\alpha\mu}$ we use $N^{\dashv\sigma\alpha}$ instead of the more conventional $N^{\alpha\sigma}$, because this latter notation would be ambiguous when using $g^{\mu\nu}$ to raise indices. While our notation differs from previous literature on the Einstein-Schrödinger theory, this change is required by our explicit metric definition, and it is necessary to be consistent with the much larger body of literature on Einstein-Maxwell theory.

2.2 The Einstein equations

To set $\delta\mathcal{L}/\delta(\sqrt{-N}N^{\dashv\mu\nu})=0$ we need some initial results. Using (2.4) and the identities $det(sM)=s^n det(M)$, $det(M^{-1})=1/det(M)$ gives

$$\sqrt{-N} = (-det(\sqrt{-N}N^{\dashv\cdot\cdot}))^{1/(n-2)}, \tag{2.24}$$

$$\sqrt{-g} = (-det(\sqrt{-g}\,g^{\cdot\cdot}))^{1/(n-2)} = (-det(\sqrt{-N}N^{\dashv(\cdot\cdot)}))^{1/(n-2)}. \tag{2.25}$$

Using (2.24,2.25,2.4) and the identity $\partial(det(M^{\cdot\cdot}))/\partial M^{\mu\nu} = M^{-1}_{\nu\mu} det(M^{\cdot\cdot})$ gives

$$\frac{\partial\sqrt{-N}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = \frac{N_{\nu\mu}}{(n-2)}, \qquad \frac{\partial\sqrt{-g}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = \frac{g_{\nu\mu}}{(n-2)}. \tag{2.26}$$

Setting $\delta\mathcal{L}/\delta(\sqrt{-N}N^{\dashv\mu\nu}) = 0$ using (2.10,2.26) gives the field equations,

$$0 = -16\pi\left[\frac{\partial\mathcal{L}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} - \left(\frac{\partial\mathcal{L}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})_{,\omega}}\right)_{,\omega}\right] \tag{2.27}$$

$$= \tilde{\mathcal{R}}_{\nu\mu} + 2A_{[\nu,\mu]}\sqrt{2}\,i\Lambda_b^{1/2} + \Lambda_b N_{\nu\mu} + \Lambda_z g_{\nu\mu} - 8\pi S_{\nu\mu}, \tag{2.28}$$

where $S_{\nu\mu}$ and the energy-momentum tensor $T_{\nu\mu}$ are defined by

$$S_{\nu\mu} \equiv 2\frac{\delta\mathcal{L}_m}{\delta(\sqrt{-N}N^{\dashv\mu\nu})} = 2\frac{\delta\mathcal{L}_m}{\delta(\sqrt{-g}g^{\mu\nu})}, \tag{2.29}$$

$$T_{\nu\mu} \equiv S_{\nu\mu} - \frac{1}{2}\,g_{\nu\mu}S^\alpha_\alpha, \qquad S_{\nu\mu} = T_{\nu\mu} - \frac{1}{(n-2)}g_{\nu\mu}T^\alpha_\alpha. \tag{2.30}$$

The second equality in (2.29) results because $\mathcal{L}_m$ in (2.2) contains only the metric $\sqrt{-g}\,g^{\mu\nu} = \sqrt{-N}N^{\dashv(\mu\nu)}$ from (2.4), and not $\sqrt{-N}N^{\dashv[\mu\nu]}$. Taking the symmetric and antisymmetric parts of (2.28) and using (2.21) gives

$$\tilde{\mathcal{R}}_{(\nu\mu)} + \Lambda_b N_{(\nu\mu)} + \Lambda_z g_{\nu\mu} = 8\pi\left(T_{\nu\mu} - \frac{1}{(n-2)}g_{\nu\mu}T^\alpha_\alpha\right), \tag{2.31}$$

$$N_{[\nu\mu]} = F_{\nu\mu}\sqrt{2}\,i\Lambda_b^{-1/2} - \tilde{\mathcal{R}}_{[\nu\mu]}\Lambda_b^{-1}. \tag{2.32}$$

Also from the curl of (2.32) we get

$$\tilde{\mathcal{R}}_{[\nu\mu,\sigma]} + \Lambda_b N_{[\nu\mu,\sigma]} = 0. \tag{2.33}$$

To put (2.31) into a form which looks more like the ordinary Einstein equations, we need some preliminary results. The definitions (2.4,2.22) of $g_{\nu\mu}$ and $f_{\nu\mu}$ can be inverted exactly to give $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$. An expansion in powers of Λ_b^{-1}

will better serve our purposes, and is derived in Appendix C,

$$N_{(\nu\mu)} \;=\; g_{\nu\mu} - 2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{2(n\!-\!2)} g_{\nu\mu} f^{\rho\sigma} f_{\sigma\rho}\right)\Lambda_b^{-1} + (f^4)\Lambda_b^{-2}\ldots \tag{2.34}$$

$$N_{[\nu\mu]} \;=\; f_{\nu\mu}\sqrt{2}\, i\Lambda_b^{-1/2} + (f^3)\Lambda_b^{-3/2}\ldots. \tag{2.35}$$

Here the notation (f^3) and (f^4) refers to terms like $f_{\nu\alpha}f^\alpha{}_\sigma f^\sigma{}_\mu$ and $f_{\nu\alpha}f^\alpha{}_\sigma f^\sigma{}_\rho f^\rho{}_\mu$. Let us consider the size of these higher order terms relative to the leading order term for worst-case fields accessible to measurement. In geometrized units an elementary charge has

$$Q_e = e\sqrt{\frac{G}{c^4}} = \sqrt{\frac{e^2}{\hbar c}\frac{G\hbar}{c^3}} = \sqrt{\alpha}\, l_P = 1.38\times 10^{-34} cm \tag{2.36}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant and $l_P = \sqrt{G\hbar/c^3}$ is the Planck length. If we assume that charged particles retain $f^1{}_0 \sim Q/r^2$ down to the smallest radii probed by high energy particle physics experiments (10^{-17}cm) we have from (2.36,2.12),

$$|f^1{}_0|^2/\Lambda_b \sim (Q_e/(10^{-17})^2)^2/\Lambda_b \sim 10^{-66}. \tag{2.37}$$

Here $|f^1{}_0|$ is assumed to be in some standard spherical or cartesian coordinate system. If an equation has a tensor term which can be neglected in one coordinate system, it can be neglected in any coordinate system, so it is only necessary to prove it in one coordinate system. The fields at 10^{-17}cm from an elementary charge would be larger than near any macroscopic charged object, and would also be larger than the strongest plane-wave fields. Therefore the higher order terms in (2.34-2.35) must be $< 10^{-66}$ of the leading order terms, so they will be completely negligible for most purposes.

In §2.4 we will calculate the connection equations resulting from $\delta\mathcal{L}/\delta\tilde\Gamma^\alpha_{\nu\mu}=0$. Solving these equations gives (2.62,2.63,2.67,2.69), which can be abbreviated as

$$\tilde\Gamma^\alpha_{(\nu\mu)} = \Gamma^\alpha_{\nu\mu}+\mathcal{O}(\Lambda_b^{-1}), \qquad \tilde\Gamma^\alpha_{[\nu\mu]}=\mathcal{O}(\Lambda_b^{-1/2}), \tag{2.38}$$

$$\tilde G_{\nu\mu} = G_{\nu\mu}+\mathcal{O}(\Lambda_b^{-1}), \qquad \tilde{\mathcal{R}}_{[\nu\mu]}=\mathcal{O}(\Lambda_b^{-1/2}), \tag{2.39}$$

where $\Gamma^\alpha_{\nu\mu}$ is the Christoffel connection (2.20), $\tilde{\mathcal{R}}_{\nu\mu}=\mathcal{R}_{\nu\mu}(\tilde\Gamma)$, $R_{\nu\mu}=R_{\nu\mu}(\Gamma)$ and

$$\tilde G_{\nu\mu} = \tilde{\mathcal{R}}_{(\nu\mu)}-\frac{1}{2}\,g_{\nu\mu}\tilde{\mathcal{R}}^\rho_\rho, \qquad G_{\nu\mu}=R_{\nu\mu}-\frac{1}{2}\,g_{\nu\mu}R. \tag{2.40}$$

In (2.39) the notation $\mathcal{O}(\Lambda_b^{-1})$ and $\mathcal{O}(\Lambda_b^{-1/2})$ indicates terms like $f^\sigma{}_{\nu;\alpha}f^\alpha{}_{\mu;\sigma}\Lambda_b^{-1}$ and $f_{[\nu\mu,\alpha];}{}^\alpha\Lambda_b^{-1/2}$.

From the antisymmetric part of the field equations (2.32) and (2.35,2.39) we get

$$f_{\nu\mu} = F_{\nu\mu}+\mathcal{O}(\Lambda_b^{-1}). \tag{2.41}$$

So $f_{\nu\mu}$ and $F_{\nu\mu}$ only differ by terms with Λ_b in the denominator, and the two become identical in the limit as $\Lambda_b\to\infty$. Combining the symmetric part of the field equations (2.31) with its contraction, and substituting (2.40,2.34,2.3)

$$\begin{aligned}
N_{(\nu\mu)}-\frac{1}{2}\,g_{\nu\mu}N^\rho_\rho &= g_{\nu\mu}-2\left(f_\nu{}^\sigma f_{\sigma\mu}-\frac{1}{2(n-2)}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right)\Lambda_b^{-1}\\
&\quad -\frac{1}{2}g_{\nu\mu}n+g_{\nu\mu}\left(f^{\rho\sigma}f_{\sigma\rho}-\frac{1}{2(n-2)}nf^{\rho\sigma}f_{\sigma\rho}\right)\Lambda_b^{-1}+(f^4)\Lambda_b^{-2}\dots\\
&= g_{\nu\mu}\left(1-\frac{n}{2}\right)-2f_\nu{}^\sigma f_{\sigma\mu}\Lambda_b^{-1}\\
&\quad +g_{\nu\mu}\left(\frac{1}{(n-2)}+1-\frac{n}{2(n-2)}\right)f^{\rho\sigma}f_{\sigma\rho}\Lambda_b^{-1}+(f^4)\Lambda_b^{-2}\dots\\
&= -2\left(f_\nu{}^\sigma f_{\sigma\mu}-\frac{1}{4}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right)\Lambda_b^{-1}-\left(\frac{n}{2}-1\right)g_{\nu\mu}+(f^4)\Lambda_b^{-2}\dots
\end{aligned}$$

gives the Einstein equations

$$\tilde{G}_{\nu\mu} = 8\pi T_{\nu\mu} - \Lambda_b\bigg(N_{(\nu\mu)} - \frac{1}{2}g_{\nu\mu}N^\rho_\rho\bigg) + \Lambda_z\left(\frac{n}{2}-1\right)g_{\nu\mu}, \tag{2.42}$$

$$= 8\pi T_{\nu\mu} + 2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{4}g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right) + \Lambda\left(\frac{n}{2}-1\right)g_{\nu\mu} + (f^4)\Lambda_b^{-1}\dots \tag{2.43}$$

From (2.29,2.30) we see that $T_{\nu\mu}$ will be the same as in ordinary general relativity, for example when we include classical hydrodynamics or spin-1/2 fields as in [49] or Appendix L. Therefore from (2.41,2.39), equation (2.43) differs from the ordinary Einstein equations only by terms with Λ_b in the denominator, and it becomes identical to the ordinary Einstein equations in the limit as $\Lambda_b\to\infty$ (with an observationally valid total Λ). In §2.4 we will examine how close the approximation is for Λ_b from (2.12).

2.3 Maxwell's equations

Setting $\delta\mathcal{L}/\delta A_\tau = 0$ and using (2.10,2.22) gives

$$0 = \frac{4\pi}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}}{\partial A_\tau} - \left(\frac{\partial\mathcal{L}}{\partial A_{\tau,\omega}}\right)_{,\omega}\right] \tag{2.44}$$

$$= \frac{\sqrt{2}\,i\Lambda_b^{1/2}}{2\sqrt{-g}}\,(\sqrt{-N}N^{\dashv[\omega\tau]})_{,\omega} - 4\pi j^\tau = \frac{(\sqrt{-g}f^{\omega\tau})_{,\omega}}{\sqrt{-g}} - 4\pi j^\tau, \tag{2.45}$$

where

$$j^\tau = \frac{-1}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}_m}{\partial A_\tau} - \left(\frac{\partial\mathcal{L}_m}{\partial A_{\tau,\omega}}\right)_{,\omega}\right]. \tag{2.46}$$

From (2.45,2.21) we get Maxwell's equations,

$$f^{\omega\tau}{}_{;\omega} = 4\pi j^\tau, \tag{2.47}$$

$$F_{[\nu\mu,\alpha]} = 0. \tag{2.48}$$

where $f_{\nu\mu} = F_{\nu\mu} + \mathcal{O}(\Lambda_b^{-1})$ from (2.41). From (2.2,2.46) we see that j^μ will be the same as in ordinary general relativity, for example when we include classical hydrodynamics or spin-1/2 fields as in [49] or Appendix L. From (2.41), we see that equations (2.47,2.48) differ from the ordinary Maxwell equations only by terms with Λ_b in the denominator, and these equations become identical to the ordinary Maxwell equations in the limit as $\Lambda_b \to \infty$. In §2.4 we will examine how close the approximation is for Λ_b from (2.12).

Because $\mathcal{L}_m$ couples to additional fields only through $g_{\mu\nu}$ and A_μ, any equations associated with additional fields will be the same as in ordinary general relativity. For example in the spin-1/2 case, setting $\delta\mathcal{L}/\delta\bar{\psi}=0$ will give the ordinary Dirac equation in curved space as in [49] or Appendix L. It would be interesting to investigate what results if one includes $f_{\mu\nu}$, $N_{\mu\nu}$ or $\tilde{\Gamma}^\alpha_{\mu\nu}$ in $\mathcal{L}_m$, although there does not appear to be any empirical reason for doing so. A continuity equation follows from (2.47) regardless of the type of source,

$$j^\rho{}_{;\rho} = \frac{1}{4\pi} f^{\tau\rho}{}_{;[\tau;\rho]} = 0. \tag{2.49}$$

Note that the covariant derivative in (2.47,2.49) is done using the Christoffel connection (2.20) formed from the symmetric metric (2.4).

2.4 The connection equations

Setting $\delta\mathcal{L}/\delta\tilde{\Gamma}^\alpha_{\nu\mu}=0$ requires some preliminary calculations. With the definition

$$\frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^\beta_{\tau\rho}} = \frac{\partial\mathcal{L}}{\partial\tilde{\Gamma}^\beta_{\tau\rho}} - \left(\frac{\partial\mathcal{L}}{\partial\tilde{\Gamma}^\beta_{\tau\rho,\omega}}\right)_{,\omega} \cdots \tag{2.50}$$

and (2.10,2.11) we can calculate,

$$
\begin{aligned}
-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^{\beta}_{\tau\rho}} &= 2\sqrt{-N}N^{\dashv\mu\nu}(\delta^\sigma_\beta\delta^\tau_\nu\delta^\rho_{[\mu|}\tilde\Gamma^\alpha_{\sigma|\alpha]}+\tilde\Gamma^\sigma_{\nu[\mu|}\delta^\alpha_\beta\delta^\tau_\sigma\delta^\rho_{|\alpha]})\\
&\quad -2(\sqrt{-N}N^{\dashv\mu\nu}\delta^\alpha_\beta\delta^\tau_\nu\delta^\rho_{[\mu}\delta^\omega_{\alpha]})_{,\omega}-(\sqrt{-N}N^{\dashv\mu\nu}\delta^\alpha_\beta\delta^\tau_\alpha\delta^\rho_{[\nu}\delta^\omega_{\mu]})_{,\omega}\\
&= -(\sqrt{-N}N^{\dashv\rho\tau})_{,\beta}-\tilde\Gamma^\rho_{\beta\mu}\sqrt{-N}N^{\dashv\mu\tau}-\tilde\Gamma^\tau_{\nu\beta}\sqrt{-N}N^{\dashv\rho\nu}+\tilde\Gamma^\alpha_{\beta\alpha}\sqrt{-N}N^{\dashv\rho\tau}\\
&\quad +\delta^\rho_\beta((\sqrt{-N}N^{\dashv\omega\tau})_{,\omega}+\tilde\Gamma^\tau_{\nu\mu}\sqrt{-N}N^{\dashv\mu\nu})+\delta^\tau_\beta(\sqrt{-N}N^{\dashv[\rho\omega]})_{,\omega}, \qquad (2.51)\\
-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^{\alpha}_{\alpha\rho}} &= (n-2)(\sqrt{-N}N^{\dashv[\rho\omega]})_{,\omega}, \qquad (2.52)\\
-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^{\alpha}_{\tau\alpha}} &= (n-1)((\sqrt{-N}N^{\dashv\omega\tau})_{,\omega}+\tilde\Gamma^\tau_{\nu\mu}\sqrt{-N}N^{\dashv\mu\nu})+(\sqrt{-N}N^{\dashv[\tau\omega]})_{,\omega}. \qquad (2.53)
\end{aligned}
$$

In these last two equations, the index contractions occur after the derivatives. At this point we must be careful. Because $\tilde\Gamma^\alpha_{\nu\mu}$ has the symmetry (2.8), it has only $n^3\!-n$ independent components, so there can only be n^3-n independent field equations associated with it. It is shown in Appendix B that instead of just setting (2.51) to zero, the field equations associated with such a field are given by the expression,

$$
\begin{aligned}
0 &= 16\pi\left[\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^{\beta}_{\tau\rho}}-\frac{\delta^\tau_\beta}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^{\alpha}_{\alpha\rho}}-\frac{\delta^\rho_\beta}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^{\alpha}_{\tau\alpha}}\right] \qquad (2.54)\\
&= (\sqrt{-N}N^{\dashv\rho\tau})_{,\beta}+\tilde\Gamma^\rho_{\beta\mu}\sqrt{-N}N^{\dashv\mu\tau}+\tilde\Gamma^\tau_{\nu\beta}\sqrt{-N}N^{\dashv\rho\nu}-\tilde\Gamma^\alpha_{\beta\alpha}\sqrt{-N}N^{\dashv\rho\tau}\\
&\quad -\delta^\tau_\beta(\sqrt{-N}N^{\dashv[\rho\omega]})_{,\omega}+\frac{1}{(n-1)}((n-2)\delta^\tau_\beta(\sqrt{-N}N^{\dashv[\rho\omega]})_{,\omega}+\delta^\rho_\beta(\sqrt{-N}N^{\dashv[\tau\omega]})_{,\omega})\\
&= (\sqrt{-N}N^{\dashv\rho\tau})_{,\beta}+\tilde\Gamma^\tau_{\nu\beta}\sqrt{-N}N^{\dashv\rho\nu}+\tilde\Gamma^\rho_{\beta\mu}\sqrt{-N}N^{\dashv\mu\tau}-\tilde\Gamma^\alpha_{\beta\alpha}\sqrt{-N}N^{\dashv\rho\tau}\\
&\qquad -\frac{1}{(n-1)}(\delta^\tau_\beta(\sqrt{-N}N^{\dashv[\rho\omega]})_{,\omega}-\delta^\rho_\beta(\sqrt{-N}N^{\dashv[\tau\omega]})_{,\omega})\\
&= (\sqrt{-N}N^{\dashv\rho\tau})_{,\beta}+\tilde\Gamma^\tau_{\sigma\beta}\sqrt{-N}N^{\dashv\rho\sigma}+\tilde\Gamma^\rho_{\beta\sigma}\sqrt{-N}N^{\dashv\sigma\tau}-\tilde\Gamma^\alpha_{\beta\alpha}\sqrt{-N}N^{\dashv\rho\tau}\\
&\qquad -\frac{8\pi\sqrt{2}\,i}{(n-1)\Lambda_b^{1/2}}\sqrt{-g}j^{[\rho}\delta^{\tau]}_\beta. \qquad (2.55)
\end{aligned}
$$

These are the connection equations, analogous to $(\sqrt{-g}g^{\rho\tau})_{;\beta}=0$ in the symmetric case. Note that we can also derive Ampere's law (2.45) by antisymmetrizing and contracting these equations. From the definition of matrix inverse $N^{\dashv\rho\tau}=(1/N)\partial N/\partial N_{\tau\rho}$, $N^{\dashv\rho\tau}N_{\tau\mu}\!=\!\delta^\rho_\mu$ we get the identity

$$(\sqrt{-N}\,)_{,\sigma}=\frac{\partial\sqrt{-N}}{\partial N_{\tau\rho}}N_{\tau\rho,\sigma}=\frac{\sqrt{-N}}{2}N^{\dashv\rho\tau}N_{\tau\rho,\sigma}=-\frac{\sqrt{-N}}{2}N^{\dashv\rho\tau}{}_{,\sigma}N_{\tau\rho}. \tag{2.56}$$

Contracting (2.55) with $N_{\tau\rho}$ using (2.8,2.56), and dividing this by $(n\!-\!2)$ gives,

$$(\sqrt{-N}\,)_{,\,\beta}-\tilde\Gamma^\alpha_{\alpha\beta}\sqrt{-N}=-\frac{8\pi\sqrt{2}\,i}{(n\!-\!1)(n\!-\!2)\Lambda_b^{1/2}}\sqrt{-g}j^\rho N_{[\rho\beta]}. \tag{2.57}$$

From (2.57) we get

$$\tilde\Gamma^\alpha_{\alpha[\nu,\mu]}-\frac{8\pi\sqrt{2}\,i}{(n\!-\!1)(n\!-\!2)\Lambda_b^{1/2}}\left(\frac{\sqrt{-g}}{\sqrt{-N}}j^\rho N_{[\rho[\nu]}\right)_{,\mu]}=(ln\sqrt{-N}\,)_{,[\nu,\mu]}=0. \tag{2.58}$$

From (2.55,2.57) we get the contravariant connection equations,

$$N^{\dashv\rho\tau}{}_{,\beta}+\tilde\Gamma^\tau_{\sigma\beta}N^{\dashv\rho\sigma}+\tilde\Gamma^\rho_{\beta\sigma}N^{\dashv\sigma\tau}=\frac{8\pi\sqrt{2}\,i}{(n\!-\!1)\Lambda_b^{1/2}}\frac{\sqrt{-g}}{\sqrt{-N}}\left(j^{[\rho}\delta^{\tau]}_\beta+\frac{1}{(n\!-\!2)}j^\alpha N_{[\alpha\beta]}N^{\dashv\rho\tau}\right). \tag{2.59}$$

Multiplying this by $-N_{\nu\rho}N_{\tau\mu}$ gives the covariant connection equations,

$$N_{\nu\mu,\beta}-\tilde\Gamma^\alpha_{\nu\beta}N_{\alpha\mu}-\tilde\Gamma^\alpha_{\beta\mu}N_{\nu\alpha}=\frac{-8\pi\sqrt{2}\,i}{(n\!-\!1)\Lambda_b^{1/2}}\frac{\sqrt{-g}}{\sqrt{-N}}\left(N_{\nu[\alpha}N_{\beta]\mu}+\frac{1}{(n\!-\!2)}N_{[\alpha\beta]}N_{\nu\mu}\right)j^\alpha. \tag{2.60}$$

Equation (2.60) together with (2.31,2.33,2.8) are often used to define the Einstein-Schrödinger theory, particularly when $T_{\nu\mu}\!=\!0$, $j^\alpha\!=\!0$.

Equations (2.55) or (2.60) can be solved exactly as in [50, 51] or §6.1, similar to the way $g_{\rho\tau;\beta}=0$ can be solved to get the Christoffel connection. An expansion in powers of Λ_b^{-1} will better serve our purposes, and such an expansion is derived in

Appendix D, and is also stated without derivation in [45],

$$\tilde{\Gamma}^{\alpha}_{\nu\mu} = \Gamma^{\alpha}_{\nu\mu} + \Upsilon^{\alpha}_{\nu\mu}, \tag{2.61}$$

$$\Upsilon^{\alpha}_{(\nu\mu)} = -2\left[f^{\tau}{}_{(\nu}f_{\mu)}{}^{\alpha}{}_{;\tau} + f^{\alpha\tau}f_{\tau(\nu;\mu)} + \frac{1}{4(n-2)}((f^{\rho\sigma}f_{\sigma\rho})_{,}{}^{\alpha}g_{\nu\mu} - 2(f^{\rho\sigma}f_{\sigma\rho})_{,(\nu}\delta^{\alpha}_{\mu)}) + \frac{4\pi}{(n-2)}j^{\rho}\left(f^{\alpha}{}_{\rho}\,g_{\nu\mu} + \frac{2}{(n-1)}f_{\rho(\nu}\delta^{\alpha}_{\mu)}\right)\right]\Lambda_b^{-1} + (f^{4\prime})\Lambda_b^{-2}\ldots, \tag{2.62}$$

$$\Upsilon^{\alpha}_{[\nu\mu]} = \left[\frac{1}{2}(f_{\nu\mu;}{}^{\alpha} + f^{\alpha}{}_{\mu;\nu} - f^{\alpha}{}_{\nu;\mu}) + \frac{8\pi}{(n-1)}j_{[\nu}\delta^{\alpha}_{\mu]}\right]\sqrt{2}\,i\Lambda_b^{-1/2} + (f^{3\prime})\Lambda_b^{-3/2}\ldots, \tag{2.63}$$

$$\Upsilon^{\alpha}_{\alpha\nu} = 2\left[\frac{1}{2(n-2)}(f^{\rho\sigma}f_{\sigma\rho})_{,\nu} - \frac{8\pi}{(n-1)(n-2)}j^{\alpha}f_{\alpha\nu}\right]\Lambda_b^{-1} + (f^{4\prime})\Lambda_b^{-2}\ldots. \tag{2.64}$$

In (2.61), $\Gamma^{\alpha}_{\nu\mu}$ is the Christoffel connection (2.20). The notation $(f^{3\prime})$ and $(f^{4\prime})$ refers to terms like $f^{\alpha}{}_{\tau}f^{\tau}{}_{\sigma}f^{\sigma}{}_{[\nu;\mu]}$ and $f^{\alpha}{}_{\tau}f^{\tau}{}_{\sigma}f^{\sigma}{}_{\rho}f^{\rho}{}_{(\nu;\mu)}$. As in (2.34,2.35), we see from (2.37) that the higher order terms in (2.62-2.64) must be $<10^{-66}$ of the leading order terms, so they will be completely negligible for most purposes.

Extracting $\Upsilon^{\tau}_{\sigma\beta}$ of (2.61) from (2.11) gives (R.6,R.7),

$$\tilde{\mathcal{R}}_{(\nu\mu)} = R_{\nu\mu} + \Upsilon^{\alpha}_{(\nu\mu);\alpha} - \Upsilon^{\alpha}_{\alpha(\nu;\mu)} - \Upsilon^{\sigma}_{(\nu\alpha)}\Upsilon^{\alpha}_{(\sigma\mu)} - \Upsilon^{\sigma}_{[\nu\alpha]}\Upsilon^{\alpha}_{[\sigma\mu]} + \Upsilon^{\sigma}_{(\nu\mu)}\Upsilon^{\alpha}_{\sigma\alpha}, \tag{2.65}$$

$$\tilde{\mathcal{R}}_{[\nu\mu]} = \Upsilon^{\alpha}_{[\nu\mu];\alpha} - \Upsilon^{\sigma}_{(\nu\alpha)}\Upsilon^{\alpha}_{[\sigma\mu]} - \Upsilon^{\sigma}_{[\nu\alpha]}\Upsilon^{\alpha}_{(\sigma\mu)} + \Upsilon^{\sigma}_{[\nu\mu]}\Upsilon^{\alpha}_{\sigma\alpha}. \tag{2.66}$$

Substituting (2.61-2.64,2.47) into (2.65) using $\ell=f^{\rho\sigma}f_{\sigma\rho}$ gives

$$
\begin{aligned}
\tilde{\mathcal{R}}_{(\nu\mu)} &= R_{\nu\mu}+\Upsilon^{\alpha}_{(\nu\mu);\,\alpha}-\Upsilon^{\alpha}_{\alpha(\nu;\mu)}-\Upsilon^{\sigma}_{[\nu\alpha]}\Upsilon^{\alpha}_{[\sigma\mu]}\dots\\
&= R_{\nu\mu}-2\Bigg[\bigg(f^{\tau}{}_{(\nu}f_{\mu)}{}^{\alpha}{}_{;\,\tau}+f^{\alpha\tau}f_{\tau(\nu;\,\mu)}+\frac{1}{4(n-2)}(\,\ell_{,}{}^{\alpha}g_{\nu\mu}-2\,\ell_{,(\nu}\delta^{\alpha}_{\mu)})\bigg)_{;\,\alpha}\\
&\qquad+\frac{4\pi}{(n-2)}j^{\rho}{}_{;\alpha}\bigg(f^{\alpha}{}_{\rho}g_{\nu\mu}+\frac{2}{(n-1)}f_{\rho(\nu}\delta^{\alpha}_{\mu)}\bigg)\\
&\qquad+\frac{4\pi}{(n-2)}j^{\rho}\bigg(f^{\alpha}{}_{\rho}g_{\nu\mu}+\frac{2}{(n-1)}f_{\rho(\nu}\delta^{\alpha}_{\mu)}\bigg)_{;\alpha}\\
&\qquad+\frac{1}{2(n-2)}\,\ell_{,(\nu;\,\mu)}-\frac{8\pi}{(n-1)(n-2)}(j^{\alpha}f_{\alpha(\nu})_{;\mu)}\\
&\qquad-\frac{1}{4}\bigg(f_{\nu\alpha;}{}^{\sigma}+f^{\sigma}{}_{\alpha;\nu}-f^{\sigma}{}_{\nu;\,\alpha}+\frac{16\pi}{(n-1)}j_{[\nu}\delta^{\sigma}_{\alpha]}\bigg)\\
&\qquad\times\bigg(f_{\sigma\mu;}{}^{\alpha}+f^{\alpha}{}_{\mu;\,\sigma}-f^{\alpha}{}_{\sigma;\,\mu}+\frac{16\pi}{(n-1)}j_{[\sigma}\delta^{\alpha}_{\mu]}\bigg)\Bigg]\Lambda_b^{-1}\dots\\
&= R_{\nu\mu}-\bigg[2f^{\tau}{}_{(\nu}f_{\mu)}{}^{\alpha}{}_{;\tau;\alpha}+2f^{\alpha\tau}f_{\tau(\nu;\,\mu);\,\alpha}+\frac{1}{2(n-2)}\,\ell_{,}{}^{\alpha}{}_{;\alpha}g_{\nu\mu}\\
&\quad-f^{\sigma}{}_{\nu;\alpha}f^{\alpha}{}_{\mu;\sigma}+f^{\sigma}{}_{\nu;\,\alpha}f_{\sigma\mu;}{}^{\alpha}+\frac{1}{2}f^{\sigma}{}_{\alpha;\nu}f^{\alpha}{}_{\sigma;\,\mu}\\
&\quad+8\pi j^{\tau}f_{\tau(\nu;\mu)}-\frac{32\pi^2}{(n-1)}j_{\nu}j_{\mu}+\frac{32\pi^2}{(n-2)}j^{\rho}j_{\rho}g_{\nu\mu}+\frac{8\pi}{(n-2)}j^{\rho}{}_{;\alpha}f^{\alpha}{}_{\rho}g_{\nu\mu}\bigg]\Lambda_b^{-1}\dots,\\
\tilde{\mathcal{R}}^{\rho}_{\rho} &= R-\bigg[2f^{\tau\beta}f_{\beta}{}^{\alpha}{}_{;\tau;\alpha}+\frac{n}{2(n-2)}\,\ell_{,}{}^{\alpha}{}_{;\alpha}-f^{\sigma\beta}{}_{;\alpha}f^{\alpha}{}_{\beta;\sigma}+\frac{1}{2}f^{\sigma\beta}{}_{;\,\alpha}f_{\sigma\beta;}{}^{\alpha}\\
&\quad-8\pi f^{\alpha\tau}j_{\tau;\alpha}-32\pi^2\bigg(1+\frac{1}{(n-1)}-\frac{n}{(n-2)}\bigg)j^{\rho}j_{\rho}+\frac{8\pi n}{(n-2)}j^{\rho}{}_{;\alpha}f^{\alpha}{}_{\rho}\bigg]\Lambda_b^{-1}\dots\\
&= R+\bigg[-2f^{\tau\beta}f_{\beta}{}^{\alpha}{}_{;\tau;\alpha}-\frac{n}{2(n-2)}\,\ell_{,}{}^{\alpha}{}_{;\alpha}-\frac{3}{2}f_{[\sigma\beta;\alpha]}f^{[\sigma\beta}{}_{;}{}^{\alpha]}\\
&\qquad-\frac{32\pi^2 n}{(n-1)(n-2)}j^{\rho}j_{\rho}-\frac{16\pi}{(n-2)}f^{\alpha\tau}j_{\tau;\alpha}\bigg]\Lambda_b^{-1}\dots.
\end{aligned}
$$

and using (2.40) gives

$$
\begin{aligned}
(\tilde{G}_{\nu\mu}-G_{\nu\mu}) &= -\bigg(2f^{\tau}{}_{(\nu}f_{\mu)}{}^{\alpha}{}_{;\tau;\alpha}+2f^{\alpha\tau}f_{\tau(\nu;\,\mu);\alpha}\\
&\qquad-f^{\sigma}{}_{\nu;\alpha}f^{\alpha}{}_{\mu;\sigma}+f^{\sigma}{}_{\nu;\alpha}f_{\sigma\mu;}{}^{\alpha}+\frac{1}{2}f^{\sigma}{}_{\alpha;\nu}f^{\alpha}{}_{\sigma;\,\mu}\\
&\qquad-g_{\nu\mu}f^{\tau\beta}f_{\beta}{}^{\alpha}{}_{;\tau;\alpha}-\frac{1}{4}g_{\nu\mu}(f^{\rho\sigma}f_{\sigma\rho})_{,}{}^{\alpha}{}_{;\alpha}-\frac{3}{4}g_{\nu\mu}f_{[\sigma\beta;\alpha]}f^{[\sigma\beta}{}_{;}{}^{\alpha]}\\
&\qquad+8\pi j^{\tau}f_{\tau(\nu;\mu)}-\frac{32\pi^2}{(n-1)}j_{\nu}j_{\mu}+\frac{16\pi^2}{(n-1)}g_{\nu\mu}j^{\rho}j_{\rho}\bigg)\Lambda_b^{-1}\dots.
\end{aligned}\tag{2.67}
$$

From (2.43) we can define an "effective" energy momentum tensor $\tilde{T}_{\nu\mu}$ which applies when $G_{\nu\mu}$ is used in the Einstein equations and $\mathcal{L}_m\!=\!0$,

$$8\pi\tilde{T}_{\nu\mu} \;=\; 2\left(f_\nu{}^\sigma f_{\sigma\mu} - \frac{1}{4}\,g_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right) - (\tilde{G}_{\nu\mu} - G_{\nu\mu}). \tag{2.68}$$

Substituting (2.63,2.47) into (2.66) gives

$$\begin{aligned}
\tilde{\mathcal{R}}_{[\nu\mu]} &= \Upsilon^\alpha_{[\nu\mu];\alpha} + \mathcal{O}(\Lambda_b^{-3/2})\ldots\\
&= \left(\frac{1}{2}(f_{\nu\mu;}{}^\alpha + f^\alpha{}_{\mu;\nu} - f^\alpha{}_{\nu;\mu})_{;\alpha} + \frac{8\pi}{(n-1)}j_{[\nu,\mu]}\right)\sqrt{2}\,i\Lambda_b^{-1/2}\ldots\\
&= \left(\frac{3}{2}f_{[\nu\mu,\alpha];}{}^\alpha + f^\alpha{}_{\mu;\nu;\alpha} - f^\alpha{}_{\nu;\mu;\alpha} + \frac{8\pi}{(n-1)}j_{[\nu,\mu]}\right)\sqrt{2}\,i\Lambda_b^{-1/2}\ldots\\
&= \left(\frac{3}{2}f_{[\nu\mu,\alpha];}{}^\alpha + 2f^\alpha{}_{\mu;[\nu;\alpha]} - 2f^\alpha{}_{\nu;[\mu;\alpha]} - \frac{8\pi(n-2)}{(n-1)}\,j_{[\nu,\mu]}\right)\sqrt{2}\,i\Lambda_b^{-1/2}\ldots.
\end{aligned} \tag{2.69}$$

As we have already noted in §2.2 and §2.3, the Λ_b in the denominator of (2.67,2.69) causes our Einstein and Maxwell equations (2.43,2.47,2.48) to become the ordinary Einstein and Maxwell equations in the limit as $\omega_c\to\infty$, $|\Lambda_z|\to\infty$, $\Lambda_b\to\infty$, and it also causes the relation $f_{\nu\mu}\approx F_{\nu\mu}$ from (2.41) to become exact in this limit. Let us examine how close these approximations are when $\Lambda_b\sim 10^{66}cm^{-2}$ as in (2.12).

We will start with the Einstein equations (2.43). Let us consider worst-case values of $\tilde{G}_{\nu\mu} - G_{\nu\mu}$ accessible to measurement, and compare these to the ordinary electromagnetic term in the Einstein equations (2.43). If we assume that charged particles retain $f^1{}_0\sim Q/r^2$ down to the smallest radii probed by high energy particle physics experiments (10^{-17}cm) we have,

$$|f^1{}_{0;1}/f^1{}_0|^2/\Lambda_b \;\sim\; 4/\Lambda_b\,(10^{-17})^2\sim 10^{-32}, \tag{2.70}$$

$$|f^1{}_{0;1;1}/f^1{}_0|/\Lambda_b \;\sim\; 6/\Lambda_b\,(10^{-17})^2\sim 10^{-32}. \tag{2.71}$$

So for electric monopole fields, terms like $f^{\sigma}{}_{\nu;\alpha}f^{\alpha}{}_{\mu;\sigma}\Lambda_b^{-1}$ and $f^{\alpha\tau}f_{\tau(\nu;\,\mu);\alpha}\Lambda_b^{-1}$ in (2.67) must be $<10^{-32}$ of the ordinary electromagnetic term in (2.43). And regarding j^{τ} as a substitute for $(1/4\pi)f^{\omega\tau}{}_{;\omega}$ from (2.47), the same is true for the j_{ν} terms. For an electromagnetic plane-wave in a flat background space we have

$$A_{\mu} = A\epsilon_{\mu}\sin(k_{\alpha}x^{\alpha}) \ , \ \epsilon^{\alpha}\epsilon_{\alpha} = -1 \ , \ k^{\alpha}k_{\alpha} = k^{\alpha}\epsilon_{\alpha} = 0, \tag{2.72}$$

$$f_{\nu\mu} = 2A_{[\mu,\nu]} = 2A\epsilon_{[\mu}k_{\nu]}\cos(k_{\alpha}x^{\alpha}), \quad j^{\sigma} = 0. \tag{2.73}$$

Here A is the magnitude, k^{α} is the wavenumber, and ϵ^{α} is the polarization. Substituting (2.72,2.73) into (2.67), all of the terms vanish for a flat background space. Also, for the highest energy gamma rays known in nature (10^{20}eV, 10^{34}Hz) we have from (2.12),

$$|f^{1}{}_{0;1}/f^{1}{}_{0}|^2/\Lambda_b \sim (E/\hbar c)^2/\Lambda_b \sim 10^{-16}, \tag{2.74}$$

$$|f^{1}{}_{0;1;1}/f^{1}{}_{0}|/\Lambda_b \sim (E/\hbar c)^2/\Lambda_b \sim 10^{-16}. \tag{2.75}$$

So for electromagnetic plane-wave fields, even if some of the extra terms in (2.67) were non-zero because of spatial curvatures, they must still be $<10^{-16}$ of the ordinary electromagnetic term in (2.68). Therefore even for the most extreme worst-case fields accessible to measurement, the extra terms in the Einstein equations (2.43) must all be $<10^{-16}$ of the ordinary electromagnetic term.

Now let us look at the approximation $f_{\nu\mu}\approx F_{\nu\mu}$ from (2.41), and Maxwell's equations (2.47,2.48). From the covariant derivative commutation rule, the cyclic identity $2R_{\nu[\tau\alpha]\mu} = R_{\nu\mu\alpha\tau}$, the definition of the Weyl tensor $C_{\nu\mu\alpha\tau}$, and the Einstein equations

$R_{\nu\mu} = -\Lambda g_{\nu\mu} + (f^2)\ldots$ from (2.43) we get

$$
\begin{aligned}
2f^{\alpha}{}_{\nu;[\mu;\alpha]} &= R^{\tau}{}_{\nu\mu\alpha}f^{\alpha}{}_{\tau} + R_{\tau}{}^{\alpha}{}_{\mu\alpha}f^{\tau}{}_{\nu} = \frac{1}{2}R_{\nu\mu\alpha\tau}f^{\alpha\tau} + R^{\tau}{}_{\mu}f_{\tau\nu} \\
&= \frac{1}{2}\left(C_{\nu\mu}{}^{\alpha\tau} + \frac{4}{(n\!-\!2)}\delta^{[\alpha}_{[\nu}R^{\tau]}_{\mu]} - \frac{2}{(n\!-\!1)(n\!-\!2)}\delta^{[\alpha}_{[\nu}\delta^{\tau]}_{\mu]}R\right)f_{\alpha\tau} - R^{\tau}{}_{\mu}f_{\nu\tau} \\
&= \frac{1}{2}f^{\alpha\tau}C_{\alpha\tau\nu\mu} + \frac{(n\!-\!2)\Lambda}{(n\!-\!1)}f_{\nu\mu} + (f^3)\ldots. \qquad (2.76)
\end{aligned}
$$

Substituting (2.35) into the antisymmetric field equations (2.32) gives

$$
f_{\nu\mu} = F_{\nu\mu} + \tilde{\mathcal{R}}_{[\nu\mu]}\sqrt{2}\,i\Lambda_b^{-1/2}/2 + (f^3)\Lambda_b^{-1}\ldots, \qquad (2.77)
$$

and using (2.69,2.76) we get

$$
f_{\nu\mu} = F_{\nu\mu} + \left(\theta_{[\tau,\alpha]}\varepsilon_{\nu\mu}{}^{\tau\alpha} + f^{\alpha\tau}C_{\alpha\tau\nu\mu} + \frac{2(n\!-\!2)\Lambda}{(n\!-\!1)}f_{\nu\mu} + \frac{8\pi(n\!-\!2)}{(n\!-\!1)}j_{[\nu,\mu]} + (f^3)\right)\Lambda_b^{-1}. \qquad (2.78)
$$

where $\varepsilon_{\tau\nu\mu\alpha} = $ (Levi$-$Civita tensor), $C_{\alpha\tau\nu\mu} = $ (Weyl tensor), and

$$
\theta_{\tau} = \frac{1}{4}f_{[\nu\mu,\alpha]}\varepsilon_{\tau}{}^{\nu\mu\alpha}, \quad f_{[\nu\mu,\alpha]} = -\frac{2}{3}\,\theta_{\tau}\varepsilon^{\tau}{}_{\nu\mu\alpha}. \qquad (2.79)
$$

The $\theta_{[\tau,\alpha]}\varepsilon_{\nu\mu}{}^{\tau\alpha}\Lambda_b^{-1}$ term in (2.78) is divergenceless so that it has no effect on Ampere's law (2.47). The $f_{\nu\mu}\Lambda/\Lambda_b$ term is $\sim 10^{-122}$ of $f_{\nu\mu}$ from (2.3,2.12). The $(f^3)\Lambda_b^{-1}$ term is $< 10^{-66}$ of $f_{\nu\mu}$ from (2.37). The largest observable values of the Weyl tensor might be expected to occur near the Schwarzschild radius, $r_s = 2Gm/c^2$, of black holes, where it takes on values around r_s/r^3. The largest value of r_s/r^3 would occur near the lightest black holes, which would be of about one solar mass, where from (2.12),

$$
\frac{C_{0101}}{\Lambda_b} \sim \frac{1}{\Lambda_b r_s^2} = \frac{1}{\Lambda_b}\left(\frac{c^2}{2Gm_\odot}\right)^2 \sim 10^{-77}. \qquad (2.80)
$$

And regarding j^{τ} as a substitute for $(1/4\pi)f^{\omega\tau}{}_{;\omega}$ from (2.47), the $j_{[\nu,\mu]}\Lambda_b^{-1}$ term is $< 10^{-32}$ of $f_{\nu\mu}$ from (2.71). Therefore, the last four terms in (2.78) must all be

$< 10^{-32}$ of $f_{\nu\mu}$. Consequently, even for the most extreme worst-case fields accessible to measurement, the extra terms in Maxwell's equations (2.47,2.48) must be $< 10^{-32}$ of the ordinary terms.

The divergenceless term $\theta_{[\tau,\alpha]}\varepsilon_{\nu\mu}{}^{\tau\alpha}\Lambda_b^{-1}$ of (2.78) should also be expected to be $< 10^{-32}$ of $f_{\nu\mu}$ from (2.70,2.71,2.79). However, we need to consider the possibility where θ_τ changes extremely rapidly. Taking the curl of (2.78), the $F_{\nu\mu}$ and $j_{[\nu,\mu]}$ terms drop out,

$$f_{[\nu\mu,\sigma]} = \left(\theta_{\tau;\alpha;[\sigma}\,\varepsilon_{\nu\mu]}{}^{\tau\alpha} + (f^{\alpha\tau}C_{\alpha\tau[\nu\mu}),_{\sigma]} + \frac{2(n-2)\Lambda}{(n-1)}f_{[\nu\mu,\sigma]} + (f^{3\prime})\right)\Lambda_b^{-1}\dots.$$

Contracting this with $\Lambda_b\varepsilon^{\rho\sigma\nu\mu}/2$ and using (2.79) gives,

$$2\Lambda_b\theta^\rho \;=\; -2\,\theta^{[\rho}{}_{;}{}^{\sigma]}{}_{;\sigma} + \frac{1}{2}\varepsilon^{\rho\sigma\nu\mu}(f^{\alpha\tau}C_{\alpha\tau[\nu\mu}),_{\sigma]} + \frac{4(n-2)\Lambda}{(n-1)}\,\theta^\rho + (f^{3\prime})\dots$$

Using $\theta^\sigma{}_{;\sigma} = 0$ from the definition (2.79), the covariant derivative commutation rule, and the Einstein equations $R_{\nu\mu} = -\Lambda g_{\nu\mu} + (f^2)\dots$ from (2.43), gives $\theta^\sigma{}_{;\rho;\sigma} = R_{\sigma\rho}\theta^\sigma = -\theta_\rho\Lambda + (f^{3\prime})\dots$, and we get something similar to the Proca equation[52, 53],

$$\theta_\rho = \left(-\,\theta_{\rho;\,\sigma;}{}^{\sigma} + \frac{1}{2}\varepsilon_\rho{}^{\sigma\nu\mu}(f^{\alpha\tau}C_{\alpha\tau[\nu\mu}),_{\sigma]} + \frac{(3n-7)\Lambda}{(n-1)}\,\theta_\rho + (f^{3\prime})\right)\frac{1}{2\Lambda_b}\dots. \qquad (2.81)$$

Here we are using a $(1,-1,-1,-1)$ metric signature. Equation (2.81) suggests that θ_ρ Proca-wave solutions might exist in this theory. Assuming that the magnitude of $C_{\alpha\tau\nu\mu}$ is roughly proportional to θ_ρ for such waves, and assuming that $f_{\mu\nu}$ goes according to (2.78) with $F_{\mu\nu} = 0$, the extra terms in (2.81) could perhaps be neglected in the weak field approximation. Using (2.81) and $\Lambda_b \approx -\Lambda_z = C_z\omega_c^4 l_P^2$ from (2.12), such Proca-wave solutions would have an extremely high minimum frequency

$$\omega_{Proca} = \sqrt{2\Lambda_b} \approx \sqrt{2C_z}\,\omega_c^2 l_P \sim 10^{43} rad/s, \qquad (2.82)$$

where the cutoff frequency ω_c and C_z come from (2.13,2.14).

There are several points to make about (2.81,2.82). 1) A particle associated with a θ_ρ field would have mass $\hbar\omega_{Proca}$, which is much greater than could be produced by particle accelerators, and so it would presumably not conflict with high energy physics experiments. 2) We have recently shown that $sin[\,kr-\omega t]$ Proca-wave solutions do not exist in the theory, using an asyptotically flat Newman-Penrose $1/r$ expansion similar to [54, 55]. However, it is still possible that wave-packet solutions could exist. 3) Substituting the $k=0$ flat space Proca-wave solution $\theta_\rho=(0,1,0,0)sin[\omega_{Proca}t]$ and $F_{\mu\nu}=0$ into (2.78,2.68,2.67), and assuming a flat background space gives $\tilde{T}_{00}=-2/\Lambda_b<0$. This suggests that Proca-wave solutions might have negative energy, but because $sin[\,kr-\omega t]$ solutions do not exist, and because of the other approximations used, this calculation is extremely uncertain. 4) With a cutoff frequency $\omega_c\sim 1/l_P$ from (2.13) we have $\omega_{Proca}>\omega_c$ from (2.82,2.13,2.14), so Proca-waves would presumably be cut off. More precisely, (2.82) says that Proca-waves would be cut off if $\omega_c>1/(l_P\sqrt{2C_z}\,)$. Whether ω_c is caused by a discreteness, uncertainty or foaminess of spacetime near the Planck length[56, 57, 58, 59, 60], or by some other effect, the same ω_c which cuts off Λ_z in (2.12) should also cut off very high frequency electromagnetic and gravitational waves, and Proca-waves. 5) If wave-packet Proca-wave solutions do exist, and they have negative energy, it is possible that θ_ρ could function as a kind of built-in Pauli-Villars field. Pauli-Villars regularization in quantum electrodynamics requires a negative energy Proca field with a mass $\hbar\omega_{Proca}$ that goes to infinity as $\omega_c\rightarrow\infty$, as we have from (2.82). This idea is supported by the effective weak field Lagrangian derived in Appendix J, and is discussed more fully in Appendix K. 6) As mentioned initially, it might be more correct to take the limit of this theory as $\omega_c \to \infty$, $|\Lambda_z| \to \infty$, $\Lambda_b \to \infty$, as in quantum electrodynamics. In this limit (2.81,2.82) require that $\theta_\rho \to 0$ or $\omega_{Proca} \to \infty$, and the theory becomes exactly Einstein-Maxwell theory as in (2.15). 7) Finally, we should emphasize that Proca-wave solutions are only a possibility suggested by equation (2.81). Their existence and their possible interpretation are just speculation at this point. We are continuing to pursue these questions.

Chapter 3

Exact Solutions

3.1 An exact electric monopole solution

Here we present an exact charged solution for this theory which closely approximates the Reissner-Nordström solution[61, 62] of Einstein-Maxwell theory. The solution is derived in Appendix N, and a MAPLE program[63] which checks the solution is available. The solution is

$$ds^2 = \check{c}adt^2 - \frac{1}{\check{c}a}dr^2 - \check{c}r^2d\theta^2 - \check{c}r^2sin^2\theta d\phi^2, \tag{3.1}$$

$$f^{10} = \frac{Q}{\check{c}r^2}, \qquad \sqrt{-N} = r^2 sin\,\theta, \qquad \sqrt{-g} = \check{c}r^2 sin\,\theta, \tag{3.2}$$

$$F_{01} = -A_0' = \frac{Q}{r^2}\left[1 + \frac{4M}{\Lambda_b r^3} - \frac{4\Lambda}{3\Lambda_b} + 2\left(\check{c} - 1 - \frac{Q^2\hat{V}}{\Lambda_b r^4}\right)\left(1 - \frac{\Lambda}{\Lambda_b}\right)\right], \tag{3.3}$$

$$a = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{Q^2\hat{V}}{r^2}\left(1 - \frac{\Lambda}{\Lambda_b}\right), \tag{3.4}$$

where ($'$) means $\partial/\partial r$, and $\check{c}$ and $\hat{V}$ are very close to one for ordinary radii,

$$\check{c} = \sqrt{1 - \frac{2Q^2}{\Lambda_b r^4}} = 1 - \frac{Q^2}{\Lambda_b r^4} \cdots - \frac{(2i)!}{[i!]^2 4^i (2i-1)}\left(\frac{2Q^2}{\Lambda_b r^4}\right)^i, \tag{3.5}$$

$$\hat{V} = \frac{r\Lambda_b}{Q^2}\left(\int r^2\check{c}\,dr - \frac{r^3}{3}\right) = 1 + \frac{Q^2}{10\Lambda_b r^4} \cdots + \frac{(2i)!}{i!(i+1)!\,4^i(4i+1)}\left(\frac{2Q^2}{\Lambda_b r^4}\right)^i, \tag{3.6}$$

and the nonzero connections are

$$\begin{aligned}
&\tilde\Gamma^1_{00} = \frac{aa'\check{c}^2}{2} - \frac{4a^2Q^2}{\Lambda_b\, r^5} \ \ , \ \tilde\Gamma^0_{10} = \tilde\Gamma^0_{01} = \frac{a'}{2a} \ \ , \ \tilde\Gamma^1_{11} = \frac{-a'}{2a},\\
&\tilde\Gamma^2_{12} = \tilde\Gamma^2_{21} = \tilde\Gamma^3_{13} = \tilde\Gamma^3_{31} = \frac{1}{r},\\
&\tilde\Gamma^1_{22} = -ar \ \ , \ \tilde\Gamma^1_{33} = -ar\sin^2\theta \ \ , \ \tilde\Gamma^3_{23} = \tilde\Gamma^3_{32} = \cot\theta \ \ , \ \tilde\Gamma^2_{33} = -\sin\theta\cos\theta,\\
&\tilde\Gamma^2_{02} = -\tilde\Gamma^2_{20} = \tilde\Gamma^3_{03} = -\tilde\Gamma^3_{30} = -\frac{a\sqrt{2}\,iQ}{\sqrt{\Lambda_b}\, r^3} \ \ , \ \tilde\Gamma^1_{10} = -\tilde\Gamma^1_{01} = -\frac{2a\sqrt{2}\,iQ}{\sqrt{\Lambda_b}\, r^3}.
\end{aligned} \tag{3.7}$$

With $\Lambda_z = 0, \Lambda_b = \Lambda$ we get the Papapetrou solution[46, 47] of the unmodified Einstein-Schrödinger theory. In this case the $M/\Lambda_b r^3$ term in (3.3) would be huge from (2.3), and the Q^2/r^2 term in (3.4) disappears, which is why the Papapetrou solution was found to be unsatisfactory[46]. However, we are instead assuming $\Lambda_b \approx -\Lambda_z$ from (2.12). In this case the solution matches the Reissner-Nordström solution except for terms which are negligible for ordinary radii. To see this, first recall that $\Lambda/\Lambda_b \sim 10^{-122}$ from (2.3,2.12), so the Λ terms are all extremely tiny. Ignoring the Λ terms and keeping only the $\mathcal{O}(\Lambda_b^{-1})$ terms in (3.3,3.4,3.5,3.6) gives

$$F_{01} = \frac{Q}{r^2}\left[1 + \frac{4M}{\Lambda_b r^3} - \frac{4Q^2}{\Lambda_b r^4}\right] + \mathcal{O}(\Lambda_b^{-2}), \tag{3.8}$$

$$A_0 = \frac{Q}{r}\left[1 + \frac{M}{\Lambda_b r^3} - \frac{4Q^2}{5\Lambda_b r^4}\right] + \mathcal{O}(\Lambda_b^{-2}), \tag{3.9}$$

$$a = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}\left[1 + \frac{Q^2}{10\Lambda_b r^4}\right] + \mathcal{O}(\Lambda_b^{-2}), \tag{3.10}$$

$$\check{c} = 1 - \frac{Q^2}{\Lambda_b r^4} + \mathcal{O}(\Lambda_b^{-2}). \tag{3.11}$$

For the smallest radii probed by high-energy particle physics we get from (2.37),

$$\frac{Q^2}{\Lambda_b r^4} \sim 10^{-66}. \tag{3.12}$$

The worst-case value of $M/\Lambda_b r^3$ might be near the Schwarzschild radius r_s of black holes where $r = r_s = 2M$ and $M/\Lambda_b r^3 = 1/2\Lambda_b r_s^2$. This value will be largest for the lightest black holes, and the lightest black hole that we can expect to observe would be of about one solar mass, where we have

$$\frac{M}{\Lambda_b r^3} \sim \frac{1}{2\Lambda_b r_s^2} = \frac{1}{2\Lambda_b}\left(\frac{c^2}{2Gm_\odot}\right)^2 \sim 10^{-77}. \tag{3.13}$$

Also, an electron has $M = Gm_e/c^2 = 7\times10^{-56}cm$, and using (2.12) and the smallest radii probed by high-energy particle physics ($10^{-17}cm$) we have

$$\frac{M}{\Lambda_b r^3} \sim \frac{7\times10^{-56}}{10^{66}(10^{-17})^3} \sim 10^{-70}. \tag{3.14}$$

From (3.12,3.13,3.14,2.3,2.12) we see that our electric monopole solution (3.1-3.4) has a fractional difference from the Reissner-Nordström solution of at most 10^{-66} for worst-case radii accessible to measurement. Clearly our solution does not have the deficiencies of the Papapetrou solution[46, 47] in the original theory, and it is almost certainly indistinguishable from the Reissner-Nordström solution experimentally. Also, from (6.142-6.147) the solution is of Petrov Type-D. And the solution reduces to the Schwarzschild solution for $Q\!=0$. And from (3.8-3.11) we see that the solution goes to the Reissner-Nordström solution exactly in the limit as $\Lambda_b\!\rightarrow\!\infty$.

The only significant difference between our electric monopole solution and the Reissner-Nordström solution occurs on the Planck scale. From (3.1,3.5), the surface area of the solution is[64],

$$\begin{pmatrix}\text{surface}\\ \text{area}\end{pmatrix} = \int_0^\pi d\theta \int_0^{2\pi} d\phi\sqrt{g_{\theta\theta}g_{\phi\phi}} = 4\pi r^2\check{c} = 4\pi r^2\sqrt{1-\frac{2Q^2}{\Lambda_b r^4}}. \tag{3.15}$$

The origin of the solution is where the surface area vanishes, so in our coordinates the origin is not at $r\!=0$ but rather at

$$r_0 = \sqrt{Q}(2/\Lambda_b)^{1/4}. \tag{3.16}$$

An alternative coordinate system is investigated in Appendix P where the origin is at $\rho=0$, but it is less satisfactory in most respects than the one we are using. From (2.36,2.12) we have $r_0\sim l_P\sim 10^{-33}cm$ for an elementary charge, and $r_0\ll 2M$ for any realistic astrophysical black hole. For $Q/M<1$ the behavior at the origin is hidden behind an event horizon nearly identical to that of the Reissner-Nordström solution. For $Q/M>1$ where there is no event horizon, the behavior at the origin differs markedly from the simple naked singularity of the Reissner-Nordström solution. For the Reissner-Nordström solution all of the relevant fields have singularities at the origin, with $g_{00}\sim Q^2/r^2$, $A_0=Q/r$, $F_{01}=Q/r^2$, $R_{00}\sim 2Q^4/r^6$ and $R_{11}\sim 2/r^2$. For our solution the metric has a less severe singularity at the origin, with $g_{11}\sim -\sqrt{r}/\sqrt{r-r_0}$. Also, the fields $N_{\mu\nu}$, $N^{\dashv\nu\mu}$, $\sqrt{-N}$, A_ν, $\sqrt{-g}f^{\nu\mu}$, $\sqrt{-g}f_{\nu\mu}$, $\sqrt{-g}g^{\nu\mu}$, $\sqrt{-g}g_{\nu\mu}$, and the functions "a" and $\hat{V}$ all have finite nonzero values and derivatives at the origin, because it can be shown (see Appendix O) that $\hat{V}(r_0)=\sqrt{2}\,[\Gamma(1/4)]^2/6\sqrt{\pi}-2/3=1.08137$. The fields $F_{\nu\mu}$, $\tilde{\Gamma}^\alpha_{\mu\nu}$ and $\sqrt{-g}\,\tilde{\mathcal{R}}_{\nu\mu}$ are also finite and nonzero at the origin, so if we use the tensor density form of the field equations (2.28,2.47), there is no ambiguity as to whether the field equations are satisfied at this location.

Finally let us consider the result from (2.37) that $|f^\mu{}_\sigma\Lambda_b^{-1/2}|<10^{-33}$ for worst-case electromagnetic fields accessible to measurement. The "smallness" of this value may seem unappealing at first, considering that $g^{\mu\nu}$ and $f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}$ are part of the total field $(\sqrt{-N}/\sqrt{-g})N^{\dashv\nu\mu}=g^{\mu\nu}+f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}$ as in (2.23). However, for an elementary charge, $|f^{\mu\nu}\Lambda_b^{-1/2}|$ is not really small if one compares it to $g^{\mu\nu}-\eta^{\mu\nu}$ instead of $g^{\mu\nu}$. Our charged solution (3.1,3.2,3.4) has $g^{00}\approx 1+2M/r+Q^2/r^2$ and $f^{01}\approx Q/r^2$. So for an elementary charge, we see from (2.36,2.12) that $|f^{01}\Lambda_b^{-1/2}|\sim Q^2/r^2$ for any radius.

3.2 An exact electromagnetic plane-wave solution

Here we present an exact electromagnetic plane-wave solution for this theory which is identical to the electromagnetic plane-wave solution in Einstein-Maxwell theory, sometimes called the Baldwin-Jeffery solution[65, 66, 67]. We will not do a full derivation, but a MAPLE program[63] which checks the solution is available. We present the solution in the form of a pp-wave solution[68], and a gravitational wave component is included for generality. The solution is expressed in terms of null coordinates $x,\ y,\ u=(t-z)/\sqrt{2},\ v=(t+z)/\sqrt{2}$,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & H & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sqrt{-g}f^{\mu\nu} = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & \check{f}_x \\ 0 & 0 & 0 & \check{f}_y \\ 0 & 0 & 0 & 0 \\ -\check{f}_x & -\check{f}_y & 0 & 0 \end{pmatrix}, \tag{3.17}$$

$$f_{\mu\nu} = 2A_{[\nu,\mu]} = 2A_{,[\nu}k_{\mu]} = \sqrt{2}\begin{pmatrix} 0 & 0 & -\check{f}_x & 0 \\ 0 & 0 & -\check{f}_y & 0 \\ \check{f}_x & \check{f}_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sqrt{-g} = \sqrt{-N} = 1, \tag{3.18}$$

where

$$k_\mu = (0,0,-1,0), \quad A_\mu = (0,0,A,0), \quad A = -\sqrt{2}(x\check{f}_x + y\check{f}_y), \tag{3.19}$$

$$H = 2\hat{H} + A^2 \tag{3.20}$$

$$= 2(h_+x^2 + h_\times xy - h_+y^2) + 2(\check{f}_x^2 + \check{f}_y^2)(x^2+y^2), \tag{3.21}$$

$$\hat{H} = h_+x^2 + h_\times xy - h_+y^2 + (y\check{f}_x - x\check{f}_y)^2, \tag{3.22}$$

and the nonzero connections are

$$\begin{aligned}
\tilde{\Gamma}^1_{33} &= \frac{1}{2}\frac{\partial H}{\partial x}, \quad \tilde{\Gamma}^2_{33} = \frac{1}{2}\frac{\partial H}{\partial y}, \quad \tilde{\Gamma}^4_{33} = \frac{1}{2}\frac{\partial H}{\partial u} - \frac{2}{\Lambda_b}\frac{\partial(\check{f}_x^2 + \check{f}_y^2)}{\partial u}, \\
\tilde{\Gamma}^4_{13} &= \frac{1}{2}\frac{\partial H}{\partial x} - \frac{2i}{\sqrt{\Lambda_b}}\frac{\partial \check{f}_x}{\partial u}, \quad \tilde{\Gamma}^4_{31} = \frac{1}{2}\frac{\partial H}{\partial x} + \frac{2i}{\sqrt{\Lambda_b}}\frac{\partial \check{f}_x}{\partial u}, \\
\tilde{\Gamma}^4_{23} &= \frac{1}{2}\frac{\partial H}{\partial y} - \frac{2i}{\sqrt{\Lambda_b}}\frac{\partial \check{f}_y}{\partial u}, \quad \tilde{\Gamma}^4_{32} = \frac{1}{2}\frac{\partial H}{\partial y} + \frac{2i}{\sqrt{\Lambda_b}}\frac{\partial \check{f}_y}{\partial u}.
\end{aligned} \tag{3.23}$$

Here $h_+(u), h_\times(u)$ characterize the gravitational wave component, $\check{f}_x(u), \check{f}_y(u)$ characterize the electromagnetic wave component, and all of these are arbitrary functions of the coordinate $u = (t-z)/\sqrt{2}$.

For the parameterization (3.17-3.20), it happens that $\tilde{\mathcal{R}}_{\mu\nu} = R_{\mu\nu}$, and the electromagnetic field is a null field[68, 66] with $f^\sigma{}_\mu f^\mu{}_\sigma = det(f^\mu{}_\nu) = 0$. For this case, as shown in §6.1, all of the higher order terms in (2.34,2.35,2.43) vanish so that $F_{\mu\nu} = f_{\mu\nu} = N_{[\mu\nu]}\Lambda_b^{1/2}/\sqrt{2}\,i$ and our Einstein and Maxwell equations are identical to those of Einstein-Maxwell theory. Maxwell's equations (2.47,2.48) are satisfied automatically from (3.18,3.17), and the Einstein equations reduce to,

$$0 = \tilde{R}_{33} + \Lambda_b(N_{33} - g_{33}) = \frac{\partial^2 \hat{H}}{\partial x^2} + \frac{\partial^2 \hat{H}}{\partial y^2} - 2(\check{f}_x^2 + \check{f}_y^2). \tag{3.24}$$

This has the solution (3.22,3.21). In Appendix Q the solution above is transformed to ordinary x, y, z, t coordinates, and also to the alternative x, y, u, v coordinates of [66]. The solution has been discussed extensively in the literature on Einstein-Maxwell theory[65, 68, 66, 67] so we will not interpret it further. It is the same solution which forms the incoming waves for the Bell-Szekeres colliding plane-wave solution[67], although the full Bell-Szekeres solution does not satisfy our theory because the electromagnetic field is not null after the collision.

Chapter 4

The equations of motion

4.1 The Lorentz force equation

A generalized contracted Bianchi identity for this theory can be derived using only the connection equations (2.55) and the symmetry (2.8) of $\tilde\Gamma^\alpha_{\nu\mu}$,

$$(\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\lambda}+\sqrt{-N}N^{\dashv\sigma\nu}\tilde{\mathcal{R}}_{\lambda\sigma})_{,\nu}-\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\nu,\lambda}=0. \tag{4.1}$$

This identity can also be written in a manifestly covariant form

$$(\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\lambda}+\sqrt{-N}N^{\dashv\sigma\nu}\tilde{\mathcal{R}}_{\lambda\sigma})_{;\nu}-\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\nu;\lambda}=0, \tag{4.2}$$

or in terms of $g^{\rho\tau}$, $f^{\rho\tau}$ and $\tilde G_{\nu\mu}$ from (2.4,2.22,2.40),

$$\tilde G^\sigma_{\nu;\,\sigma}=\left(\frac{3}{2}f^{\sigma\rho}\,\tilde{\mathcal{R}}_{[\sigma\rho,\nu]}+f^{\alpha\sigma}{}_{;\alpha}\tilde{\mathcal{R}}_{[\sigma\nu]}\right)\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{4.3}$$

The identity was originally derived[3, 7] assuming $j^\nu=0$ in (2.55), and later expressed in terms of the metric (2.4) by [23, 43, 44, 37]. The derivation for $j^\nu\!\neq\!0$ was first done[45] by applying an infinitesimal coordinate transformation to an invariant integral, and it is also done in Appendix E using a much different direct computation method. Clearly (4.1,4.3) are generalizations of the ordinary contracted Bianchi identity $2(\sqrt{-g}\,R^{\nu}{}_{\lambda})_{,\nu}-\sqrt{-g}\,g^{\nu\sigma}R_{\sigma\nu,\lambda}=0$ or $G^{\sigma}_{\nu;\sigma}=0$, which is also valid in this theory.

Another useful identity[23] is derived in Appendix A using only the definitions (2.4,2.22) of $g_{\mu\nu}$ and $f_{\mu\nu}$,

$$\Big(N^{(\mu}{}_{\nu)}-\frac{1}{2}\delta^{\mu}_{\nu}N^{\rho}_{\rho}\Big)_{;\,\mu}=\Big(\frac{3}{2}f^{\sigma\rho}N_{[\sigma\rho,\nu]}+f^{\sigma\rho}{}_{;\sigma}N_{[\rho\nu]}\Big)\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{4.4}$$

The ordinary Lorentz force equation results from taking the divergence of the Einstein equations (2.42) using (4.3,2.47,2.32,4.4,2.21)

$$\begin{aligned}
8\pi T^{\sigma}_{\nu;\,\sigma} &= \Big(\frac{3}{2}f^{\sigma\rho}\,\tilde{\mathcal{R}}_{[\sigma\rho,\nu]}+4\pi j^{\sigma}\tilde{\mathcal{R}}_{[\sigma\nu]}\Big)\sqrt{2}\,i\Lambda_b^{-1/2}+\Lambda_b\Big(N^{(\mu}{}_{\nu)}-\frac{1}{2}\delta^{\mu}_{\nu}N^{\rho}_{\rho}\Big)_{;\,\mu} && (4.5)\\
&= \Big(4\pi j^{\sigma}\tilde{\mathcal{R}}_{[\sigma\nu]}-\Lambda_b\frac{3}{2}f^{\sigma\rho}N_{[\sigma\rho,\nu]}\Big)\sqrt{2}\,i\Lambda_b^{-1/2}+\Lambda_b\Big(N^{(\mu}{}_{\nu)}-\frac{1}{2}\delta^{\mu}_{\nu}N^{\rho}_{\rho}\Big)_{;\,\mu} && (4.6)\\
&= (4\pi j^{\sigma}\tilde{\mathcal{R}}_{[\sigma\nu]}+\Lambda_b f^{\rho\sigma}{}_{;\rho}N_{[\sigma\nu]})\sqrt{2}\,i\Lambda_b^{-1/2} && (4.7)\\
&= 4\pi j^{\sigma}(\tilde{\mathcal{R}}_{[\sigma\nu]}+\Lambda_b N_{[\sigma\nu]})\sqrt{2}\,i\Lambda_b^{-1/2} && (4.8)\\
&= 16\pi j^{\sigma}A_{[\sigma,\nu]}, && (4.9)\\
T^{\sigma}_{\nu;\,\sigma} &= F_{\nu\sigma}j^{\sigma}. && (4.10)
\end{aligned}$$

See Appendix H for an alternative derivation of the Lorentz force equation. In Appendix L we also show that the Lorentz force equation and the continuity equation can be derived from the Klein-Gordon equation for spin-0 fields. Note that the covariant derivatives in (4.2,4.3,4.4,4.10) are all done using the Christoffel connection (2.20) formed from the symmetric metric (2.4).

4.2 Equations of motion of the electric monopole solution

Here we calculate the equations of motion when one body is much heavier than the other so this body remains approximately stationary and is represented by the charged solution (3.1-3.7). We ignore radiation reaction effects. The Lorentz-force equation (4.10) for the classical hydrodynamics case is

$$Q_2 F^{\alpha}{}_{\mu} u^{\mu} = \frac{du^{\alpha}}{d\lambda} + \Gamma^{\alpha}_{\mu\nu} u^{\mu} u^{\nu}, \quad u^{\nu} = \frac{dx^{\nu}}{d\lambda}. \tag{4.11}$$

The stationary and moving bodies have masses M, M_2 and charges Q, Q_2. We are using $d\lambda = ds/M_2$ instead of ds because the unitless parameter λ is still meaningful for the null geodesics of photons where $ds \to 0$ and $M_2 \to 0$. Using the metric (3.1) and the relation $(r^2\check{c})' = 2r/\check{c}$ from (3.5), the non-zero Christoffel connections (2.20) are

$$\Gamma^1_{00} = \frac{a\check{c}}{2}(a\check{c})' \;,\; \Gamma^0_{10} = \frac{(a\check{c})'}{2a\check{c}} \;,\; \Gamma^1_{11} = -\frac{(a\check{c})'}{2a\check{c}} \;,\; \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{\check{c}^2 r}, \tag{4.12}$$
$$\Gamma^1_{22} = -ar \;,\; \Gamma^1_{33} = -ar\sin^2\theta \;,\; \Gamma^3_{23} = \cot\theta \;,\; \Gamma^2_{33} = -\sin\theta\cos\theta.$$

The equations of motion (4.11) are then

$$a\check{c} Q_2 F_{01} u^t = \frac{du^r}{d\lambda} - \frac{(a\check{c})'}{2a\check{c}} u^{r2} - aru^{\theta 2} - arsin^2\theta u^{\phi 2} + \frac{a\check{c}(a\check{c})'}{2} u^{t2}, \tag{4.13}$$

$$0 = \frac{du^{\theta}}{d\lambda} + \frac{2}{r\check{c}^2} u^r u^{\theta} - \sin\theta cos\theta\, u^{\phi 2}, \tag{4.14}$$

$$0 = \frac{du^{\phi}}{d\lambda} + \frac{(r^2\check{c})'}{r^2\check{c}}\, u^r u^{\phi} + 2cot\theta\, u^{\theta} u^{\phi}, \tag{4.15}$$

$$\frac{Q_2 F_{01}}{a\check{c}}\, u^r = \frac{du^t}{d\lambda} + \frac{(a\check{c})'}{a\check{c}} u^r u^t. \tag{4.16}$$

For motion in the equatorial plane we may put $u^\theta = 0,\ \theta = \pi/2$, and (4.14) is identically satisfied. Then from (4.15) we get

$$0 = \frac{1}{r^2\check{c}}\frac{d(u^\phi r^2\check{c})}{d\lambda}, \tag{4.17}$$

$$u^\phi r^2\check{c} = (constant) = L = (angular\ momentum). \tag{4.18}$$

From (4.16,3.3) we get

$$0 = \frac{1}{a\check{c}}\left(\frac{d(u^t a\check{c})}{d\lambda} + Q_2 A_0' u^r\right) = \frac{1}{a\check{c}}\frac{d}{d\lambda}\left(u^t a\check{c} + Q_2 A_0\right), \tag{4.19}$$

$$u^t a\check{c} + Q_2 A_0 = (constant) = E = (total\ energy). \tag{4.20}$$

Recalling that $d\lambda = ds/M_2$ and $u^\theta = 0,\ \theta = \pi/2$ we also have

$$M_2^2 = u^\alpha u_\alpha = a\check{c}u^{t2} - \frac{1}{a\check{c}}u^{r2} - \check{c}r^2 u^{\phi 2}. \tag{4.21}$$

Eliminating t and λ from (4.21) using (4.18,4.20) gives

$$M_2^2 a\check{c} = (a\check{c})^2 u^{t2} - \left(\frac{dr}{d\phi}u^\phi\right)^2 - a\check{c}^2 r^2 u^{\phi 2} = (E - Q_2 A_0)^2 - \left(\frac{dr}{d\phi}\frac{L}{r^2\check{c}}\right)^2 - \frac{aL^2}{r^2}. \tag{4.22}$$

This can be rewritten as an integral

$$\phi = \int \frac{L dr/r^2}{\check{c}\sqrt{(E - Q_2 A_0)^2 - aL^2/r^2 - M_2^2 a\check{c}}}. \tag{4.23}$$

For $\Lambda_b \to \infty,\ a = 1$ we have flat-space electrodynamics, and the integral can be done analytically. For $\Lambda_b \to \infty$ we have Einstein-Maxwell theory, and the integral becomes an elliptic integral. For a finite Λ_b the integral is more complicated, but using (3.9,3.10,3.11) for $A_0,\ a,\ \check{c}$ and neglecting powers higher than $1/r^4$ also leads to an elliptic integral. The time dependence can be obtained using (4.18,4.20) to get

$$dt/d\phi = u^t/u^\phi = (r^2\check{c}/L)(E - Q_2 A_0)/a\check{c} = (E - Q_2 A_0)r^2/aL \tag{4.24}$$

so that from (4.23),

$$t = \int \frac{(E - Q_2 A_0)dr}{a\check{c}\sqrt{(E - Q_2 A_0)^2 - aL^2/r^2 - M_2^2 a\check{c}}} \tag{4.25}$$

We can also obtain the results (4.23,4.25) using the Hamilton-Jacobi approach as in [75], p. 94-95 and 306-308. From (4.21,3.1), the Hamilton-Jacobi equation is

$$M_2^2 = g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} + Q_2 A_\mu\right)\left(\frac{\partial S}{\partial x^\nu} + Q_2 A_\nu\right) \tag{4.26}$$

$$= \frac{1}{a\check{c}}\left(\frac{\partial S}{\partial t} + Q_2 A_0\right)^2 - a\check{c}\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{\check{c}r^2}\left(\frac{\partial S}{\partial \phi}\right)^2. \tag{4.27}$$

The solution is

$$S = -Et + L\phi + S_r(r), \qquad S_r = \int \frac{dr}{a\check{c}}\sqrt{(E - Q_2 A_0)^2 - aL^2/r^2 - M_2^2 a\check{c}}. \tag{4.28}$$

Then (4.23) is obtained from the equation $\partial S/\partial L = (constant)$, and (4.25) is obtained from the equation $\partial S/\partial E = (constant)$.

Let us analyze the special case $L = Q_2 = 0$ using the effective potential method of [69, 66]. From (4.22,4.18) and the definitions $\tilde{E} = E/M_2$, $ds = M_2 d\lambda$ we have

$$a\check{c} = \tilde{E}^2 - \left(\frac{dr}{ds}\right)^2. \tag{4.29}$$

This equation can be expressed as a non-relativistic potential problem,

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 = \frac{\tilde{E}^2 - 1}{2} - \tilde{V}, \tag{4.30}$$

where $(dr/ds)^2/2$ corresponds to the kinetic energy per mass, and $\tilde{V}$ is the so-called "effective potential",

$$\tilde{V} = \frac{a\check{c} - 1}{2}. \tag{4.31}$$

Since $\hat{V}(r_0) = \sqrt{2}\,[\Gamma(1/4)]^2/6\sqrt{\pi} - 2/3 = 1.08137$, we can assume that $\hat{V} \approx 1$ for present purposes, and the effective potential becomes,

$$\tilde{V} \approx \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\sqrt{1 - \frac{2Q^2}{\Lambda_b r^4}} - \frac{1}{2}. \tag{4.32}$$

Let us consider the case for elementary particles where $Q \gg M$. This case is more interesting than astronomical objects because there is no event horizon to hide the behavior close to the origin at $r_0 = (2Q^2/\Lambda_b)^{1/4} = 3.16 \times 10^{-34} cm$ where $\check{c} = 0$. Assuming an electron charge and mass we have $Q = Q_e = e\sqrt{G/c^4} = \sqrt{\alpha}\, l_P = 1.38 \times 10^{-34} cm$ and $M = Gm_e/c^2 = 7 \times 10^{-56} cm$. In this case the mass term in "a" is smaller that the charge term for $r < Q^2/2M = 1.36 \times 10^{-13} cm$, which is close to the classical electron radius. The following table shows the rough behavior of $\tilde{V}$,

$\tilde{V}$ vs. r for three Q/Q_e values

	Our charged solution			Reissner-Nordström solution			
$r/Q_e \backslash Q/Q_e$	.265	1.06	1.86	.265	1.06	1.86	
1.68	−.05602	−	−	.01250	.20000	.61250	
2.66	−.00519	−.15032	−	.00500	.08000	.24500	
3.76	−.00002	−.00478	−.05315	.00250	.04000	.12250	
4.60	.00055	.00769	.01525	.00166	.02666	.08166	
5.32	.00062	.00953	.02611	.00125	.02000	.06125	(4.33)
5.94	.00060	.00937	.02722	.00100	.01600	.04900	
10^{21}	−.00001	.00000	.00001	−.00001	.00000	.00001	
10^{25}	< 0	< 0	< 0	< 0	< 0	< 0	
∞	.00000	.00000	.00000	.00000	.00000	.00000	

For $Q = Q_e$ we find that $\tilde{V}$ has a zero at $r_0/Q_e = (2/\Lambda_b Q_e^2)^{1/4} = 2.3$, then it rises quickly to a maximum of $\sim .00953$ at $r/Q_e = 5.3$, then it falls slowly to 0 near the classical electron radius calculated above, and it remains slightly below 0 as $r \to \infty$ where it goes to 0. Radii where $\tilde{V} = (\tilde{E} - 1)/2$ are turning points where the radial motion reverses, so a falling body would bounce back only if $(\tilde{E} - 1)/2 < .00953$.

It is the $\check{c}$ term that causes $\tilde{V}$ to have a zero at $r_0/Q_e = 2.3$, which limits the potential for small radii. Particles falling into the Reissner-Nordström solution with $L=0,\ Q \gg M,\ \check{c}=1$ would always bounce back near $r \sim Q_e \sim l_P$ regardless of their energy, because the potential goes to infinity as $r \to 0$. This is a clear difference between our charged solution and the Reissner-Nordström solution, although it is unclear whether it has any significance from an experimental viewpoint.

Now let us consider a massless particle with $M_2 = L = Q_2 = 0$ falling into a body with $Q \gg M$. Setting $M_2 = L = Q_2 = 0$ in (4.25) gives

$$t = \int \frac{dr}{a\check{c}} = \int \frac{dr}{(1 - 2M/r + Q^2/r^2)\sqrt{1 - 2Q^2/\Lambda_b r^4}}, \qquad dr = a\check{c}dt. \tag{4.34}$$

Here the $\check{c}$ term causes a pole at $r_0 = (2Q_e^2/\Lambda_b)^{1/4} = 3.16 \times 10^{-34} cm$. Ignoring the "a" term, one gets an elliptic integral which evaluates to some finite value when the lower limit is set to r_0. This means that a particle would take a finite time to reach the radius r_0, at which point it presumably disappears. For the Reissner-Nordström solution with $\check{c}=1$, the integrand becomes $\sim r^2/Q^2$ near $r=0$, so a particle would take a finite time to reach $r=0$. Therefore, in contrast to the massive case, a massless neutral particle of any energy will fall into the singularity in a finite time for either our charged solution or the Reissner-Nordström solution.

4.3 The Einstein-Infeld-Hoffmann equations of motion

Here we derive the Lorentz force from the theory using the Einstein-Infeld-Hoffmann (EIH) method[70]. For Einstein-Maxwell theory, the EIH method allows the equations of motion to be derived directly from the electro-vac field equations. For neutral particles the method has been verified to Post-Newtonian order[70], and in fact it was the method first used to derive the Post-Newtonian equations of motion[71]. For charged particles the method has been verified to Post-Coulombian order[72, 73, 74], meaning that it gives the same result as the Darwin Lagrangian[53, 75] (see also Appendix F). The EIH method is valuable because it does not require any additional assumptions, such as the postulate that neutral particles follow geodesics, or the *ad hoc* inclusion of matter terms in the Lagrangian density. When the EIH method was applied to the original Einstein-Schrödinger theory, no Lorentz force was found between charged particles[9, 10]. The basic difference between our case and [9, 10] is that our Einstein equations (2.43) contain the familiar term $f_\nu{}^\sigma f_{\sigma\mu} - (1/4) g_{\nu\mu} f^{\rho\sigma} f_{\sigma\rho}$. This term appears because we assumed $\Lambda_b \neq 0$, $\Lambda_z \neq 0$, and because of our metric definition (2.4) and (2.34). With this term, the EIH method predicts the same Lorentz force as it does for Einstein-Maxwell theory. Also, it happens that the extra terms in our approximate Einstein and Maxwell equations due to (2.67,2.78) cause no contribution beyond the Lorentz force, to Newtonian/Coulombian order. The basic reason for the null result of [9, 10] is that they assumed $\Lambda_b = 0$ and $g_{\mu\nu} = N_{(\mu\nu)}$, so that every term in their effective energy-momentum tensor has "extra derivatives"[76]. For the same reason that [9, 10] found no Lorentz force, the extra derivative terms in our effective energy-momentum tensor (2.68,2.67) cause no contribution to the equations of motion.

In §4.1 we derived the ordinary Lorentz force equation by including source terms in our theory, and taking the divergence of the Einstein equations. Here we derive the Lorentz force using the EIH method because it requires no assumptions about source terms, and also to show definitely that the well known negative result of [9, 10] for the unmodified Einstein-Schrödinger theory does not apply to our theory. We will only cover the bare essentials of the EIH method which are necessary to derive the Lorentz force, and the references above should be consulted for a more complete explanation. We will also only calculate the equations of motion to Newtonian/Coulombian order, because this is the order where the Lorentz force first appears.

With the EIH method, one does not just find equations of motion, but rather one finds approximate solutions $g_{\mu\nu}$ and $f_{\mu\nu}$ of the field equations which correspond to a system of two or more particles. These approximate solutions will in general contain $1/r^p$ singularities, and these are considered to represent particles. It happens that acceptable solutions to the field equations can only be found if the motions of these singularities are constrained to obey certain equations of motion. The assumption is that these approximate solutions for $g_{\mu\nu}$ and $f_{\mu\nu}$ should approach exact solutions asymptotically, and therefore the motions of the singularities should approximate the motions of exact solutions. Any event horizon or other unusual feature of exact solutions at small radii is irrelevant because the singularities are assumed to be separated by much larger distances, and because the method relies greatly on surface integrals done at large distances from the singularities. Some kind of exact Reissner-

Nordström-like solution should probably exist in order for the EIH method to make sense, and the charged solution (3.1-3.7) fills this role in our case. However, exact solutions are really only used indirectly to identify constants of integration.

The EIH method assumes the "slow motion approximation", meaning that $v/c \ll 1$. The fields are expanded in the form[70, 72, 73, 74],

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} - \eta_{\mu\nu}\eta^{\sigma\rho}\gamma_{\sigma\rho}/2, \tag{4.35}$$

$$\gamma_{00} = {}_2\gamma_{00}\lambda^2 + {}_4\gamma_{00}\lambda^4 \dots \tag{4.36}$$

$$\gamma_{0k} = {}_3\gamma_{0k}\lambda^3 + {}_5\gamma_{0k}\lambda^5 \dots \tag{4.37}$$

$$\gamma_{ik} = {}_4\gamma_{ik}\lambda^4 \dots \tag{4.38}$$

$$A_0 = {}_2A_0\lambda^2 + {}_4A_0\lambda^4 \dots \tag{4.39}$$

$$A_k = {}_3A_k\lambda^3 + {}_5A_k\lambda^5 \dots \tag{4.40}$$

$$f_{0k} = {}_2f_{0k}\lambda^2 + {}_4f_{0k}\lambda^4 \dots \tag{4.41}$$

$$f_{ik} = {}_3f_{ik}\lambda^3 + {}_5f_{ik}\lambda^5 \dots \tag{4.42}$$

where $\lambda \sim v/c$ is the expansion parameter, the order of each term is indicated with a left subscript[9], $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and Latin indices run from 1-3. The field $\gamma_{\mu\nu}$ (often called $\bar{h}_{\mu\nu}$ in other contexts) is used instead of $g_{\mu\nu}$ only because it simplifies the calculations. Because $\lambda \sim v/c$, when the expansions are substituted into the Einstein and Maxwell equations, a time derivative counts the same as one higher order in λ. The general procedure is to substitute the expansions, and solve the resulting field equations order by order in λ, continuing to higher orders until a desired level of accuracy is achieved. At each order in λ, one of the ${}_l\gamma_{\mu\nu}$ terms and one of the ${}_lf_{\mu\nu}$ terms will be unknowns, and the equations will involve known results from previous orders because of the nonlinearity of the Einstein equations.

The expansions (4.36-4.42) use only alternate powers of λ essentially because the Einstein and Maxwell equations are second order differential equations[71], although for higher powers of λ, all terms must be included to predict radiation[72, 73, 74]. Because $\lambda \sim v/c$, the expansions have the magnetic components A_k and f_{ik} due to motion at one order higher in λ than the electric components A_0 and f_{0i}. As in [72, 73, 74], f_{0k} and f_{ik} have even and odd powers of λ respectively. This is the opposite of [9, 10] because we are assuming a direct definition of the electromagnetic field (2.22,2.35,2.78,2.21) instead of the dual definition $f^{\alpha\rho} = \varepsilon^{\alpha\rho\sigma\mu} N_{[\sigma\mu]}/2$ assumed in [9, 10].

The field equations are assumed to be of the standard form

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad \text{where} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta}, \tag{4.43}$$

or equivalently

$$R_{\mu\nu} = 8\pi S_{\mu\nu} \quad \text{where} \quad S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta}. \tag{4.44}$$

However, with the EIH method we must solve a sort of quasi-Einstein equations,

$$0 = \breve{G}_{\mu\nu} - 8\pi \breve{T}_{\mu\nu}, \tag{4.45}$$

where

$$\breve{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta} R_{\alpha\beta}, \qquad \breve{T}_{\mu\nu} = S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta} S_{\alpha\beta}. \tag{4.46}$$

Here the use of $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$ is not an approximation because (4.44) implies (4.45) whether $\breve{G}_{\mu\nu}$ and $\breve{T}_{\mu\nu}$ are defined with $\eta_{\mu\nu}$ or $g_{\mu\nu}$. Note that the references use many different notations in (4.45): instead of $\breve{G}_{\mu\nu}$ others use $\Pi_{\mu\nu}/2{+}\Lambda_{\mu\nu}$, $\Phi_{\mu\nu}/2{+}\Lambda_{\mu\nu}$ or [LS$:\mu\nu$] and instead of $8\pi\breve{T}_{\mu\nu}$ others use $-2S_{\mu\nu}$, $-\Lambda'_{\mu\nu}$, $-\Lambda_{\mu\nu}$ or [RS$:\mu\nu$].

The equations of motion result as a condition that the field equations (4.45) have acceptable solutions. In the language of the EIH method, acceptable solutions are those that contain only "pole" terms and no "dipole" terms, and this can be viewed as a requirement that the solutions should resemble Reissner-Nordström solutions asymptotically. To express the condition of solvability we must consider the integral of the field equations (4.45) over 2D surfaces S surrounding each singularity,

$$ {}_lC_\mu = \frac{1}{2\pi}\int^S ({}_l\breve{G}_{\mu k} - 8\pi\,{}_l\breve{T}_{\mu k})n_k dS. \tag{4.47} $$

Here n_k is the surface normal and l is the order in λ. Assuming that the divergence of the Einstein equations (4.43) vanishes, and that (4.45) has been solved to all previous orders, it can be shown[70] that in the current order

$$ ({}_l\breve{G}_{\mu k} - 8\pi\,{}_l\breve{T}_{\mu k})_{|k} = 0. \tag{4.48} $$

Here and throughout this section "|" represents ordinary derivative[70]. From Green's theorem, (4.48) implies that ${}_lC_\mu$ in (4.47) will be independent of surface size and shape[70]. The condition for the existence of an acceptable solution for ${}_4\gamma_{ik}$ is simply

$$ {}_4C_i = 0, \tag{4.49} $$

and these are also our three $\mathcal{O}(\lambda^4)$ equations of motion[70]. The C_0 component of (4.47) causes no constraint on the motion[70] so we only need to calculate $\breve{G}_{ik}$ and $\breve{T}_{ik}$.

At this point let us introduce a Lemma from [70] which is derived from Stokes's theorem. This Lemma states that

$$\int^{S} \mathcal{F}_{(\cdots)kl|l} n_k dS = 0 \quad \text{if} \quad \mathcal{F}_{(\cdots)kl} = -\mathcal{F}_{(\cdots)lk}, \tag{4.50}$$

where $\mathcal{F}_{(\cdots)kl}$ is any antisymmetric function of the coordinates, n_k is the surface normal, and S is any closed 2D surface which may surround a singularity. The equation ${}_4C_i = 0$ is a condition for the existence of a solution for ${}_4\gamma_{ik}$ because ${}_4\gamma_{ik}$ is found by solving the $\mathcal{O}(\lambda^4)$ field equations (4.45), and ${}_4C_i$ is the integral (4.47) of these equations. However, because of the Lemma (4.50), it happens that the ${}_4\gamma_{ik}$ terms in ${}_4\breve{G}_{ik}$ integrate to zero in (4.47), so that ${}_4C_i$ is actually independent of ${}_4\gamma_{ik}$. In fact it is a general rule that C_i for one order can be calculated using only results from previous orders[70], and this is a crucial aspect of the EIH method. Therefore, the calculation of the $\mathcal{O}(\lambda^4)$ equations of motion (4.49) does not involve the calculation of ${}_4\gamma_{ik}$, and we will see below that it also does not involve the calculation of ${}_3f_{ik}$ or ${}_4f_{0k}$.

The ${}_4\breve{G}_{ik}$ contribution to (4.47) is derived in [70]. For two particles with masses m_1, m_2 and positions ξ_1^i, ξ_2^i, the $\mathcal{O}(\lambda^4)$ term from the integral over the first particle is

$$\overset{\breve{G}}{\underset{4}{}}C_i \;=\; \frac{1}{2\pi}\int^{1} {}_4\breve{G}_{ik} n_k dS = -4\left\{m_1\ddot{\xi}_1^i - m_1 m_2 \frac{\partial}{\partial \xi_1^i}\left(\frac{1}{r}\right)\right\}, \tag{4.51}$$

where

$$r \;=\; \sqrt{(\xi_1^s - \xi_2^s)(\xi_1^s - \xi_2^s)}\,. \tag{4.52}$$

If there is no other contribution to (4.47), then (4.49) requires that $\overset{\breve{G}}{\underset{4}{}}C_i = 0$ in (4.51), and the particle acceleration will be proportional to a $\nabla(m_1 m_2/r)$ Newtonian gravitational force. These are the EIH equations of motion for vacuum general relativity to $\mathcal{O}(\lambda^4)$, or Newtonian order.

Because our effective energy-momentum tensor (2.68) is quadratic in $f_{\mu\nu}$, and the expansions (4.36-4.42) begin with λ^2 terms, the $\mathcal{O}(\lambda^2)-\mathcal{O}(\lambda^3)$ calculations leading to (4.51) are unaffected by the addition of the electromagnetic terms to the vacuum field equations. However, the $8\pi\,{}_4\breve{T}_{ik}$ contribution to (4.47) will add to the ${}_4\breve{G}_{ik}$ contribution. To calculate this contribution, we will assume that our singularities in $f_{\nu\mu}$ are simple moving Coulomb potentials, and that $\theta^\rho=0$, $\Lambda=0$. Then from (2.78,4.41-4.42) we see that ${}_2F_{0k}={}_2f_{0k}$, and from inspection of the extra terms in our Maxwell equations (2.47,2.48,2.78) and Proca equation (2.81), we see that these equations are both solved to $\mathcal{O}(\lambda^3)$. Because (2.68) is quadratic in $f_{\mu\nu}$, we see from (4.41-4.42) that only ${}_2f_{0k}$ can affect the $\mathcal{O}(\lambda^4)$ equations of motion. Including only ${}_2f_{0k}$, our $f_{\mu\nu}$ is then a sum of two Coulomb potentials with charges Q_1, Q_2 and positions ξ^i_1, ξ^i_2 of the form

$$
\begin{aligned}
{}_2A_\mu &= ({}_2\varphi,0,0,0) \quad , \quad {}_2f_{0k}=2\,{}_2A_{[k|0]}=-\,{}_2\varphi_{|k}, &(4.53)\\
{}_2\varphi &= \psi^1+\psi^2 \quad , \quad \psi^1=Q_1/r_1 \quad , \quad \psi^2=Q_2/r_2, &(4.54)\\
r_p &= \sqrt{(x^s-\xi^s_p)(x^s-\xi^s_p)} \quad , \quad p=1...2\,. &(4.55)
\end{aligned}
$$

Because (2.68) is quadratic in both $f_{\mu\nu}$ and $g_{\mu\nu}$, and the expansions (4.36-4.42) start at λ^2 in both of these quantitites, no gravitational-electromagnetic interactions will occur at $\mathcal{O}(\lambda^4)$. This allows us to replace covariant derivatives with ordinary derivatives, and $g_{\nu\mu}$ with $\eta_{\nu\mu}$ in (2.68). This also allows us to replace $\breve{T}_{\mu\nu}$ from

(4.45,4.46) with the effective energy momentum tensor from (2.68),

$$
\begin{aligned}
8\pi\breve{T}_{\nu\mu} &= 2\left(f_\nu{}^\sigma f_{\sigma\mu}-\frac{1}{4}\eta_{\nu\mu}f^{\rho\sigma}f_{\sigma\rho}\right)\\
&+\Bigg(2f^\tau{}_{(\nu}f_{\mu)}{}^\alpha{}_{|\tau|\alpha}+2f^{\alpha\tau}f_{\tau(\nu|\mu)|\alpha}-f^\sigma{}_{\nu|\alpha}f^\alpha{}_{\mu|\sigma}+f^\sigma{}_{\nu|\alpha}f_{\sigma\mu,}{}^\alpha+\frac{1}{2}f^\sigma{}_{\alpha|\nu}f^\alpha{}_{\sigma|\mu}\\
&-\eta_{\nu\mu}f^{\tau\beta}f_\beta{}^\alpha{}_{|\tau|\alpha}-\frac{1}{4}\eta_{\nu\mu}(f^{\rho\sigma}f_{\sigma\rho})_|{}^\alpha{}_{|\alpha}-\frac{3}{4}\eta_{\nu\mu}f_{[\sigma\beta|\alpha]}f^{[\sigma\beta}{}_|{}^{\alpha]}+(f^4)\Bigg)\Lambda_b^{-1}. \quad (4.56)
\end{aligned}
$$

This can be simplified by keeping only $\mathcal{O}(\lambda^4)$ terms. The terms $2f^\tau{}_{(\nu}f_{\mu)}{}^\alpha{}_{|\tau|\alpha}$ and $-\eta_{\nu\mu}f^{\tau\beta}f_\beta{}^\alpha{}_{|\tau|\alpha}$ vanish because (4.53) satisfies Ampere's law to $\mathcal{O}(\lambda^2)$. The term $-(3/4)\eta_{\nu\mu}f_{[\sigma\beta|\alpha]}f^{[\sigma\beta}{}_|{}^{\alpha]}$ vanishes because (4.53) satisfies $f_{[\sigma\beta|\alpha]}=2A_{[\beta|\sigma|\alpha]}=0$. Also, since time derivatives count the same as a higher order in λ, we can remove the term $-f^\sigma{}_{s|\alpha}f^\alpha{}_{m|\sigma}=-f^0{}_{s|0}f^0{}_{m|0}$, and we can change some of the summations over Greek indices to summations over Latin indices. The (f^4) term will be $\mathcal{O}(\lambda^8)$ so it can obviously be eliminated. And as mentioned above, only ${}_2f_{0k}$ contributes at $\mathcal{O}(\lambda^4)$. Applying these results, and dropping the order subscripts to reduce the clutter, the spatial part of (4.56) becomes,

$$
\begin{aligned}
8\pi\,{}_4\breve{T}_{sm} &= 2\left(f_s{}^0 f_{0m}-\frac{1}{2}\eta_{sm}f^{r0}f_{0r}\right)\\
&+\left(2f^{a0}f_{0(s|m)|a}+f^0{}_{s|a}f_{0m|}{}^a+f^0{}_{a|s}f^a{}_{0|m}-\frac{1}{2}\eta_{sm}(f^{r0}f_{0r})_|{}^a{}_{|a}\right)\Lambda_b^{-1} \quad (4.57)\\
&= -2\left(f_{0s}f_{0m}+\frac{1}{2}\eta_{sm}f_{0r}f_{0r}\right)\\
&+\left(2f_{0a}f_{0(s|m)|a}-f_{0s|a}f_{0m|a}+f_{0a|s}f_{0a|m}+\frac{1}{2}\eta_{sm}(f_{0r}f_{0r})_{|a|a}\right)\Lambda_b^{-1}. \quad (4.58)
\end{aligned}
$$

Note that ${}_2\varphi$ from (4.54) obeys Gauss's law,

$$\varphi_{|a|a}=0. \quad (4.59)$$

Substituting (4.53) into (4.58) and using (4.59) gives

$$
\begin{aligned}
8\pi\,{}_4\breve{T}_{sm} &= -2\left(\varphi_{|s}\varphi_{|m} + \frac{1}{2}\eta_{sm}\varphi_{|r}\varphi_{|r}\right) \\
&\quad + \left(2\varphi_{|a}\varphi_{|s|m|a} - \varphi_{|s|a}\varphi_{|m|a} + \varphi_{|a|s}\varphi_{|a|m} + \frac{1}{2}\eta_{sm}(\varphi_{|r}\varphi_{|r})_{|a|a}\right)\Lambda_b^{-1} \quad (4.60) \\
&= -2\left(\varphi_{|s}\varphi_{|m} + \frac{1}{2}\eta_{sm}\varphi_{|r}\varphi_{|r}\right) \\
&\quad -(\varphi_{|s}\varphi_{|a|m} + \varphi_{|r}\varphi_{|r|s}\eta_{am})_{|a}\Lambda_b^{-1} + (\varphi_{|a}\varphi_{|s|m} + \varphi_{|r}\varphi_{|r|a}\eta_{sm})_{|a}\Lambda_b^{-1} \\
&= -2\left(\varphi_{|s}\varphi_{|m} + \frac{1}{2}\eta_{sm}\varphi_{|r}\varphi_{|r}\right) - 2(\varphi_{|[s}\varphi_{|a]|m} + \varphi_{|r}\varphi_{|r|[s}\eta_{a]m})_{|a}\Lambda_b^{-1}. \quad (4.61)
\end{aligned}
$$

From (4.50), the second group of terms in (4.61) integrates to zero in (4.47), so it can have no effect on the equations of motion. The first group of terms in (4.61) is what one gets with Einstein-Maxwell theory[72, 73, 74], so at this stage we have effectively proven that the theory predicts a Lorentz force.

For completeness we will finish the derivation. First, we see from (4.61,4.59) that ${}_4\breve{T}_{sm|s} = 0$. This is to be expected because of (4.48), and it means that the $8\pi\,{}_4\breve{T}_{sm}$ contribution to the surface integral (4.47) will be independent of surface size and shape. This also means that only 1/distance^2 terms such as η_{sm}/r^2 or $x_s x_m/r^4$ can contribute to (4.47). The integral over a term with any other distance-dependence would depend on the surface radius, and therefore we know beforehand that it must vanish or cancel with other similar terms[70]. Now, $\varphi_{|i} = \psi^1_{|i} + \psi^2_{|i}$ from (4.54). Because $\psi^1_{|i}$ and $\psi^2_{|i}$ both go as 1/distance^2, but are in different locations, it is clear from (4.61) that contributions can only come from cross terms between the two. Including only these terms gives,

$$
8\pi\,{}_4\breve{T}^c_{sm} = -2\left(\psi^1_{|s}\psi^2_{|m} + \psi^2_{|s}\psi^1_{|m} + \eta_{sm}\psi^1_{|r}\psi^2_{|r}\right). \quad (4.62)
$$

Some integrals we will need can be found in [70]. With $\psi = 1/\sqrt{x^s x^s}$ we have,

$$\frac{1}{4\pi}\int^0 \psi_{|m} n_m dS = -1 \quad , \quad \frac{1}{4\pi}\int^0 \psi_{|a} n_m dS = -\frac{1}{3}\delta_{am}. \tag{4.63}$$

Using (4.62,4.63,4.54) and integrating over the first particle we get,

$$\frac{1}{2\pi}\int^1 \left[-8\pi\breve{T}_{sm}\right] n_m dS = \frac{1}{2\pi}\int^1 2\big(\psi^1_{|s}\psi^2_{|m} + \psi^2_{|s}\psi^1_{|m} + \eta_{sm}\psi^1_{|r}\psi^2_{|r}\big) n_m dS \tag{4.64}$$

$$= 4Q_1\psi^2_{|s}(\xi_1)\left(-\frac{1}{3} - 1 + \frac{1}{3}\right) = -4Q_1\psi^2_{|s}(\xi_1). \tag{4.65}$$

Using (4.49,4.47,4.65,4.51,4.54) we get

$$0 = {}_4C_i = -4\left\{m_1\ddot{\xi}^i_1 - m_1 m_2 \frac{\partial}{\partial \xi^i_1}\left(\frac{1}{r}\right)\right\} - 4Q_1\psi^2_{|i}(\xi_1) \tag{4.66}$$

$$= -4\left\{m_1\ddot{\xi}^i_1 - m_1 m_2 \frac{\partial}{\partial \xi^i_1}\left(\frac{1}{r}\right)\right\} - 4Q_1\frac{\partial}{\partial \xi^i_1}\left(\frac{Q_2}{r}\right)$$

$$= -4\left\{m_1\ddot{\xi}^i_1 - m_1 m_2 \frac{\partial}{\partial \xi^i_1}\left(\frac{1}{r}\right) + Q_1 Q_2 \frac{\partial}{\partial \xi^i_1}\left(\frac{1}{r}\right)\right\}, \tag{4.67}$$

where

$$r = \sqrt{(\xi^s_1 - \xi^s_2)(\xi^s_1 - \xi^s_2)}. \tag{4.68}$$

These are the EIH equations of motion for this theory to $\mathcal{O}(\lambda^4)$, or Newtonian/ Coulombian order. These equations of motion clearly exhibit the Lorentz force, and in fact they match the $\mathcal{O}(\lambda^4)$ equations of motion of Einstein-Maxwell theory.

Chapter 5

Observational consequences

5.1 Pericenter advance

Here we calculate the pericenter advance for a body with mass and charge M_2, Q_2 rotating around a more massive body with mass and charge M, Q, which is represented by the charged solution (3.1-3.7). We will use the effective potential method of [69, 66], together with the Lorentz force equation (4.10,4.11) and the resulting equations of motion calculated in §4.2. Using (4.22,4.18) and the definitions

$$\tilde{Q}_2 = Q_2/M_2, \quad \tilde{L} = L/M_2, \quad \tilde{E} = E/M_2, \quad ds = M_2 d\lambda, \tag{5.1}$$

we have

$$a\check{c} = (\tilde{E} - \tilde{Q}_2 A_0)^2 - \left(\frac{dr}{ds}\right)^2 - \frac{a\tilde{L}^2}{r^2}. \tag{5.2}$$

This equation can be expressed in the form of a non-relativistic potential problem,

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 = \frac{\tilde{E}^2 - 1}{2} - \tilde{V}, \tag{5.3}$$

where $(dr/ds)^2/2$ corresponds to the kinetic energy per mass, and $\tilde{V}$ is the so-called "effective potential",

$$\tilde{V} = -\frac{1}{2}(\tilde{E} - \tilde{Q}_2 A_0)^2 + \frac{a\tilde{L}^2}{2r^2} + \frac{a\check{c}}{2} + \frac{(\tilde{E}^2 - 1)}{2}. \tag{5.4}$$

Using the expressions (3.9,3.10,3.11) for $A_0,\ a,\ \check{c}$ and keeping only terms which fall off as $1/r^4$ or slower we get

$$\tilde{V} \approx \tilde{E}\tilde{Q}_2\left(\frac{Q}{r}+\frac{QM}{\Lambda_b r^4}\right)-\frac{\tilde{Q}_2^2 Q^2}{2r^2}+\frac{\tilde{L}^2}{2r^2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)+\frac{1}{2}\left(-\frac{2M}{r}+\frac{Q^2}{r^2}-\frac{Q^2}{\Lambda_b r^4}\right). \tag{5.5}$$

Combining the powers of $1/r$ gives

$$\tilde{V} = -\frac{\mu}{r} + \overbrace{\frac{(\tilde{L}^2 - \tilde{Q}_2^2 Q^2 + Q^2)}{2r^2}}^{\tilde{V}_{SR}\quad \tilde{V}_{RN1}} - \overbrace{\frac{M\tilde{L}^2}{r^3}}^{\tilde{V}_{GR}} + \overbrace{\frac{(Q^2\tilde{L}^2 + Q(2\tilde{E}\tilde{Q}_2 M - Q)/\Lambda_b)}{2r^4}}^{\tilde{V}_{RN2}\qquad \tilde{V}_{ES}}, \tag{5.6}$$

where

$$\mu = M - \tilde{E}\tilde{Q}_2 Q. \tag{5.7}$$

Here $-\mu/r$ is the combined Newtonian/Coulombian potential, and the term $\tilde{L}^2/2r^2$ is sometimes called the "centrifugal potential energy". These terms characterize the nonrelativistic Newtonian/Coulombian central force problem, and the associated orbits will be ellipses with a fixed pericenter. The additional terms $\tilde{V}_{SR}$, $\tilde{V}_{GR}$, $\tilde{V}_{RN1}$, $\tilde{V}_{RN2}$, $\tilde{V}_{ES}$ are due respectively to special relativity, general relativity, the Q^2/r^2 term of the Reissner-Nördstrom solution, and finally from our theory. All of these additional terms are small relative to the first terms for ordinary radii, and they can be treated as perturbations of the Newtonian/Coulombian case. Setting $d\tilde{V}/dr = 0$ gives the radius "r_0" of a stable circular orbit. If a body is displaced slightly from r_0 it will oscillate in radius about r_0, executing simple harmonic motion with proper time radial frequency ω_r given by

$$\omega_r = \sqrt{[d^2\tilde{V}/dr^2]_{r=r_0}} \,. \tag{5.8}$$

Subtracting this from the proper time angular frequency from (4.18)

$$\omega_\phi = \tilde{L}/\check{c}r^2 \tag{5.9}$$

gives the pericenter advance,

$$\omega_p = \omega_\phi - \omega_r. \tag{5.10}$$

The derivatives of the effective potential are

$$\frac{\partial \tilde{V}}{\partial r} = \frac{\mu}{r^2} - \frac{(\tilde{L}^2 - \tilde{Q}_2^2 Q^2 + Q^2)}{r^3} + \frac{3M\tilde{L}^2}{r^4} - \frac{2(Q^2\tilde{L}^2 + Q(2\tilde{E}\tilde{Q}_2 M - Q)/\Lambda_b)}{r^5}, \tag{5.11}$$

$$\frac{\partial^2 \tilde{V}}{\partial r^2} = -\frac{2\mu}{r^3} + \frac{3(\tilde{L}^2 - \tilde{Q}_2^2 Q^2 + Q^2)}{r^4} - \frac{12M\tilde{L}^2}{r^5} + \frac{10(Q^2\tilde{L}^2 + Q(2\tilde{E}\tilde{Q}_2 M - Q)/\Lambda_b)}{r^6}. \tag{5.12}$$

For the Newtonian/Coulombian case we have

$$0 = \frac{\partial \tilde{V}}{\partial r} = \frac{\mu}{r^2} - \frac{\tilde{L}^2}{r^3} = \frac{1}{r^3}(\mu r - \tilde{L}^2) \tag{5.13}$$

which has the solution

$$r_0 = \tilde{L}^2/\mu \tag{5.14}$$

$$\omega_r = \sqrt{\frac{d^2\tilde{V}}{dr^2}} = \sqrt{-\frac{2\mu}{r_0^3} + \frac{3\tilde{L}^2}{r_0^4}} = \frac{\tilde{L}}{r_0^2}. \tag{5.15}$$

From (4.18) the orbital frequency using proper time is

$$\omega_\phi = u^\phi = \frac{\tilde{L}}{r_0^2}. \tag{5.16}$$

So for the Newtonian/Coulombian case there is no pericenter advance

$$\omega_p = \omega_\phi - \omega_r = 0. \tag{5.17}$$

Each of the additional potential terms will have the uninteresting effect of changing the dependence of the orbital parameters on the constants $\tilde{L}$ and $\tilde{E}$. However these terms will also have the more fundamental effect of introducing pericenter advance. Here we will calculate the pericenter advance for each of the additional potential terms, and compare these to the results to our theory.

Including the special relativistic term V_{SR} gives

$$0 = \frac{\partial \tilde{V}}{\partial r} = \frac{\mu}{r^2} - \frac{\tilde{L}^2}{r^3} + \frac{\tilde{Q}_2^2 Q^2}{r^3} = \frac{1}{r^3}(\mu r - \tilde{L}^2 + \tilde{Q}_2^2 Q^2), \tag{5.18}$$

which has the solution

$$r_0 = (\tilde{L}^2 - \tilde{Q}_2^2 Q^2)/\mu, \tag{5.19}$$

$$\omega_r = \sqrt{\frac{d^2\tilde{V}}{dr^2}} = \sqrt{-\frac{2\mu}{r_0^3} + \frac{3(\tilde{L}^2 - \tilde{Q}_2^2 Q^2)}{r_0^4}} = \frac{\tilde{L}}{r_0^2}\sqrt{(1 - \tilde{Q}_2^2 Q^2/\tilde{L}^2)} \tag{5.20}$$

$$= \omega_\phi \left(1 - \frac{\tilde{Q}_2^2 Q^2}{2\tilde{L}^2}\right). \tag{5.21}$$

So the pericenter advance caused by the special relativistic term V_{SR} is

$$\omega_{pSR} = \omega_\phi - \omega_r = \frac{\tilde{Q}_2^2 Q^2 \omega_\phi}{2\tilde{L}^2}. \tag{5.22}$$

Including the general relativistic term V_{GR} gives

$$0 = \frac{\partial \tilde{V}}{\partial r} = \frac{\mu}{r^2} - \frac{\tilde{L}^2}{r^3} + \frac{3M\tilde{L}^2}{r^4} = \frac{1}{r^4}(\mu r^2 - \tilde{L}^2 r + 3M\tilde{L}^2). \tag{5.23}$$

This equation could potentially be solved by using the solution of a quadratic equation. However, we will instead find an approximation based on a perturbation of the

Newtonian case. We assume that

$$r_{GR} = \tilde{L}^2/\mu + \Delta r_0. \tag{5.24}$$

Assuming that Δr_0 is small compared to $\tilde{L}^2/\mu$ we can make the approximation $(\tilde{L}^2/\mu + \Delta r_0)^n \approx (\tilde{L}^2/\mu + n\Delta r_0)(\tilde{L}^2/\mu)^{n-1}$. Substituting into (5.23) gives

$$0 = \mu(\tilde{L}^2/\mu + 2\Delta r_0)\tilde{L}^2/\mu - \tilde{L}^2(\tilde{L}^2/\mu + \Delta r_0) + 3M\tilde{L}^2 = \tilde{L}^2\Delta r_0 + 3M\tilde{L}^2, \tag{5.25}$$

which has the solution

$$\begin{aligned}
\Delta r_0 &= -3M, &(5.26)\\
\omega_r &= \sqrt{\frac{d^2\tilde{V}}{dr^2}} = \sqrt{-\frac{2\mu}{r_0^3} + \frac{3\tilde{L}^2}{r_0^4} - \frac{12M\tilde{L}^2}{r_0^5}} = \frac{1}{r_0^2}\sqrt{-2\mu r_0 + 3\tilde{L}^2 - 12M\tilde{L}^2/r_0} &(5.27)\\
&= \frac{1}{r_0^2}\sqrt{-2\mu(\tilde{L}^2/\mu - 3M) + 3\tilde{L}^2 - 12M\tilde{L}^2(\tilde{L}^2/\mu + 3M)(\tilde{L}^2/\mu)^{-2}} &(5.28)\\
&= \frac{\tilde{L}}{r_0^2}\sqrt{1 - 6\mu M/\tilde{L}^2 - 36M^2\mu^2/\tilde{L}^4} &(5.29)\\
&= \omega_\phi\left(1 - \frac{3\mu M}{\tilde{L}^2}\right). &(5.30)
\end{aligned}$$

So the pericenter advance caused by the general relativistic term V_{GR} is

$$\omega_{pGR} = \omega_\phi - \omega_r = \frac{3\mu M\omega_\phi}{\tilde{L}^2}. \tag{5.31}$$

The pericenter advance caused by the 1st Reissner-Nordström term V_{RN1} can be derived from the calculations for the V_{SR} term by letting $\tilde{Q}_2^2 Q^2 \to -Q^2$,

$$\omega_{pRN1} = \omega_\phi - \omega_r = -\frac{Q^2\omega_\phi}{2\tilde{L}^2}. \tag{5.32}$$

Including the 2nd Reissner-Nordström term V_{RN2} gives

$$0 = \frac{\partial\tilde{V}}{\partial r} = \frac{\mu}{r^2} - \frac{\tilde{L}^2}{r^3} - \frac{2Q^2\tilde{L}^2}{r^5} = \frac{1}{r^5}\left(\mu r^3 - \tilde{L}^2 r^2 - 2Q^2\tilde{L}^2\right). \tag{5.33}$$

Again we assume that

$$r_{ES} = \tilde{L}^2/\mu + \Delta r_0. \tag{5.34}$$

Substituting into (5.33) and using $(\tilde{L}^2/\mu + \Delta r_0)^n \approx (\tilde{L}^2/\mu + n\Delta r_0)(\tilde{L}^2/\mu)^{n-1}$ gives

$$\begin{aligned} 0 &= \mu(\tilde{L}^2/\mu + 3\Delta r_0)(\tilde{L}^2/\mu)^2 - \tilde{L}^2(\tilde{L}^2/\mu + 2\Delta r_0)\tilde{L}^2/\mu - 2Q^2\tilde{L}^2 & (5.35)\\ &= \Delta r_0 \tilde{L}^4/\mu - 2Q^2\tilde{L}^2, & (5.36) \end{aligned}$$

which has the solution

$$\begin{aligned} \Delta r_0 &= 2\mu Q^2/\tilde{L}^2, & (5.37)\\ \omega_r &= \sqrt{\frac{d^2\tilde{V}}{dr^2}} = \sqrt{-\frac{2\mu}{r_0^3} + \frac{3\tilde{L}^2}{r_0^4} + \frac{10Q^2\tilde{L}^2}{r_0^6}} = \frac{1}{r_0^2}\sqrt{-2\mu r_0 + 3\tilde{L}^2 + \frac{10Q^2\tilde{L}^2}{r_0^2}} & (5.38)\\ &= \frac{1}{r_0^2}\sqrt{-2\mu(\tilde{L}^2/\mu + \Delta r_0) + 3\tilde{L}^2 + 10Q^2\tilde{L}^2(\tilde{L}^2/\mu - 2\Delta r_0)(\tilde{L}^2/\mu)^{-3}} & (5.39)\\ &= \frac{1}{r_0^2}\sqrt{-4\mu^2Q^2/\tilde{L}^2 + \tilde{L}^2 + 10Q^2\mu^2/\tilde{L}^2 - 40Q^4\mu^4/\tilde{L}^6} & (5.40)\\ &= \frac{\tilde{L}}{r_0^2}\sqrt{1 + 6\mu^2Q^2/\tilde{L}^4 - 40Q^4\mu^4/\tilde{L}^8} \approx \frac{\tilde{L}}{r_0^2}\left(1 + \frac{3Q^2\mu^2}{\tilde{L}^4}\right) & (5.41)\\ &= \omega_\phi\left(1 + \frac{3Q^2\mu^2}{\tilde{L}^4}\right). & (5.42) \end{aligned}$$

So the pericenter advance caused by the 2nd Reissner-Nordström term V_{RN2} is

$$\omega_{pRN2} = \omega_\phi - \omega_r = -\frac{3Q^2\mu^2\omega_\phi}{\tilde{L}^4}. \tag{5.43}$$

The pericenter advance caused by the Einstein-Schrödinger term V_{ES} can be derived from the calculations for the 2nd Reissner-Nordström term V_{RN2} by letting $Q^2\tilde{L}^2 \rightarrow Q(2\tilde{E}\tilde{Q}_2M - Q)/\Lambda_b$, except that $\omega_\phi = \tilde{L}/\check{r}r^2$ from (5.9) instead of $\omega_\phi = \tilde{L}/r^2$ from (5.16). Ignoring the correction to $r_0 = \tilde{L}^2/\mu$ from (5.14) we have

$$\omega_\phi = \frac{\tilde{L}}{r_0^2\sqrt{1 - 2Q^2/\Lambda_b r_0^4}} = \frac{\tilde{L}}{r_0^2\sqrt{1 - 2Q^2\mu^4/\Lambda_b\tilde{L}^8}} \approx \frac{\tilde{L}}{r_0^2}\left(1 + \frac{Q^2\mu^4}{\Lambda_b\tilde{L}^8}\right), \tag{5.44}$$

and from (5.41),

$$\omega_r = \frac{\tilde{L}}{r_0^2}\left(1+\frac{3Q(2\tilde{E}\tilde{Q}_2M-Q)\mu^2}{\Lambda_b\tilde{L}^6}\right) = \frac{\tilde{L}}{r_0^2}\left[1-\left(3-\frac{6\tilde{E}\tilde{Q}_2M}{Q}\right)\frac{Q^2\mu^2}{\Lambda_b\tilde{L}^6}\right]. \quad (5.45)$$

So the total pericenter advance caused by the Einstein-Schrödinger term V_{ES} is

$$\omega_{pES} = \omega_\phi - \omega_r = \left(3-\frac{6\tilde{E}\tilde{Q}_2M}{Q}+\frac{\mu^2}{\tilde{L}^2}\right)\frac{Q^2\mu^2\omega_\phi}{\Lambda_b\tilde{L}^6}. \quad (5.46)$$

Combining all of the calculations, the total pericenter advance comes to

$$\frac{\omega_p}{\omega_\phi} = \overbrace{\frac{\tilde{Q}_2^2Q^2}{2\tilde{L}^2}}^{\omega_{pSR}} + \overbrace{\frac{3\mu M}{\tilde{L}^2}}^{\omega_{pGR}} - \overbrace{\frac{Q^2}{2\tilde{L}^2}}^{\omega_{pRN1}} - \overbrace{\frac{3Q^2\mu^2}{\tilde{L}^4}}^{\omega_{pRN2}} + \overbrace{\left(3-\frac{6\tilde{E}\tilde{Q}_2M}{Q}+\frac{\mu^2}{\tilde{L}^2}\right)\frac{Q^2\mu^2}{\tilde{L}^6\Lambda_b}}^{\omega_{pES}}. \quad (5.47)$$

Here the special relativity term ω_{pSR} agrees with [75, 72, 77], the general relativity term ω_{pGR} agrees with [69, 78], and the first Reissner-Nördstrom term ω_{pRN1} agrees with [79, 80]. The ω_{pES} term is due to our theory. All of these calculations were done by assuming nearly circular orbits. However the ω_{pGR} result can be shown to be correct for arbitrary eccentricity[69] if we replace $\omega_\phi = \tilde{L}/r^2$ from (5.16) with the more general expression from Newtonian mechanics

$$\omega_\phi = \frac{\tilde{L}^2}{\sqrt{\mu}\,(1-e_s^2)a_s^{5/2}}, \quad (5.48)$$

$$e_s = (eccentricity), \quad (5.49)$$

$$a_s = (semimajor\ axis). \quad (5.50)$$

This also true of ω_{pSR}, as can be seen from p.94 of [75], and it is probably true for the other ω_p results as well. Also, using $\omega_\phi = \tilde{L}/r^2$ from (5.16) and (5.48) we can reproduce the result (5.14) for nearly circular orbits

$$\tilde{L} \approx \sqrt{\mu r}\,. \quad (5.51)$$

For a first test case we choose the Bohr atom with $M = M_P,\ M_2 = M_e,\ Q = -Q_2 = |Q_e|$ because of its approximate physical relevance, and because the Einstein-Schrödinger term will have the greatest effect at small radii. Using $M^{(geom)} = M^{(cgs)}G/c^2$, $Q^{(geom)} = Q^{(cgs)}\sqrt{G/c^4}$ and $esu = cm^{3/2}g^{1/2}/s$ we have

$$M = M_P = 1.67\times 10^{-24}g\left(\frac{6.67\times 10^{-8}cm^3/g\cdot s^2}{(3\times 10^{10}cm/s)^2}\right) = 1.24\times 10^{-52}cm, \tag{5.52}$$

$$M_2 = M_e = 9.11\times 10^{-28}g\left(\frac{6.67\times 10^{-8}cm^3/g\cdot s^2}{(3\times 10^{10}cm/s)^2}\right) = 6.75\times 10^{-56}cm, \tag{5.53}$$

$$Q = -Q_2 = |Q_e| = 4.8\times 10^{-10}esu\sqrt{\frac{6.67\times 10^{-8}cm^3/g\cdot s^2}{(3\times 10^{10}cm/s)^4}} = 1.38\times 10^{-34}cm, \tag{5.54}$$

$$r = a_0 = .529\times 10^{-8}cm, \tag{5.55}$$

$$\Lambda_b = 10^{66}cm^{-2}, \tag{5.56}$$

$$\tilde{E} = (total\ energy)/M_2 \approx 1, \tag{5.57}$$

$$\mu = -\frac{QQ_2\tilde{E}}{M_2} = \frac{(1.38\times 10^{-34}cm)^2\times 1}{6.75\times 10^{-56}cm} = 2.81\times 10^{-13}cm, \tag{5.58}$$

$$\tilde{L} = \sqrt{\mu a_0} = \sqrt{2.81\times 10^{-13}cm\times .529\times 10^{-8}cm} = 3.86\times 10^{-11}cm, \tag{5.59}$$

$$\omega_\phi = \frac{c\tilde{L}}{r^2} = \frac{3\times 10^{10}cm/s\times 3.86\times 10^{-11}cm}{(.529\times 10^{-8}cm)^2} = 4.14\times 10^{16}rad/s, \tag{5.60}$$

and

$$\frac{\omega_{pSR}}{\omega_\phi} = \frac{(1.38\times 10^{-34}cm)^4}{2(3.86\times 10^{-11}cm)^2(6.75\times 10^{-56}cm)^2} = 2.65\times 10^{-5}, \tag{5.61}$$

$$\frac{\omega_{pGR}}{\omega_\phi} = \frac{3\times 2.81\times 10^{-13}cm\times 1.24\times 10^{-52}cm}{(3.86\times 10^{-11}cm)^2} = 7.00\times 10^{-44}, \tag{5.62}$$

$$\frac{\omega_{pRN1}}{\omega_\phi} = -\frac{(1.38\times 10^{-34}cm)^2}{2(3.86\times 10^{-11}cm)^2} = -6.36\times 10^{-48}, \tag{5.63}$$

$$\frac{\omega_{pRN2}}{\omega_\phi} = -\frac{3(1.38\times 10^{-34}cm)^2(2.81\times 10^{-13}cm)^2}{(3.86\times 10^{-11}cm)^4} = -2.02\times 10^{-51}, \tag{5.64}$$

$$\frac{\omega_{pES}}{\omega_\phi} = \frac{6(1.38\times 10^{-34}cm)^2(2.81\times 10^{-13}cm)^2\times 1.24\times 10^{-52}cm}{(3.86\times 10^{-11}cm)^6\times 6.75\times 10^{-56}cm\times 10^{66}cm^{-2}} = 5.0\times 10^{-93}. \tag{5.65}$$

From (5.47,5.9), the pericenter advances are

$$\omega_{pSR} = 2.65 \times 10^{-5}\omega_\phi = 1.10 \times 10^{12} rad/s, \tag{5.66}$$

$$\omega_{pGR} = 7.00 \times 10^{-44}\omega_\phi = 2.90 \times 10^{-27} rad/s, \tag{5.67}$$

$$\omega_{pRN1} = -6.36 \times 10^{-48}\omega_\phi = -2.63 \times 10^{-31} rad/s, \tag{5.68}$$

$$\omega_{pRN2} = -2.02 \times 10^{-51}\omega_\phi = -8.36 \times 10^{-35} rad/s, \tag{5.69}$$

$$\omega_{pES} = 5.0 \times 10^{-93}\omega_\phi = 2.07 \times 10^{-76} rad/s. \tag{5.70}$$

For a second test case we choose $M = M_\odot$ because this is the smallest black hole we can expect to observe, and the smallest black hole will create the worst-case observable spatial curvature. We choose an extremal black hole with $Q = M$ because this is the worst-case charge which avoids a naked singularity, and we choose the orbital radius to be $r = 4M$ because this is close to the smallest stable orbit. For the second body we choose $Q_2 = 0$ and $M_2 << M$, so that M_2 does not enter into the equations. Using $M^{(geom)} = M^{(cgs)} G/c^2$ we have

$$\mu = M = Q = M_\odot = 1.99 \times 10^{33} g \left(\frac{6.67 \times 10^{-8} cm^3/g \cdot s^2}{(3 \times 10^{10} cm/s)^2} \right) = 1.47 \times 10^5 cm, \tag{5.71}$$

$$r = 4M = 5.90 \times 10^5 cm, \tag{5.72}$$

$$Q_2 = 0 cm, \tag{5.73}$$

$$\Lambda_b = 10^{66} cm^{-2}, \tag{5.74}$$

$$\tilde{E} = (total\ energy)/M_2 \approx 1, \tag{5.75}$$

$$\tilde{L} = \sqrt{\mu r} = \sqrt{1.47 \times 10^5 cm \times 5.90 \times 10^5 cm} = 2.95 \times 10^5 cm, \tag{5.76}$$

$$\omega_\phi = \frac{c\tilde{L}}{r^2} = \frac{3 \times 10^{10} cm/s \times 2.95 \times 10^5 cm}{(5.90 \times 10^5 cm)^2} = 2.55 \times 10^4 rad/s, \tag{5.77}$$

and

$$\frac{\omega_{pSR}}{\omega_\phi} = 0, \tag{5.78}$$

$$\frac{\omega_{pGR}}{\omega_\phi} = \frac{3\times 1.47\times 10^5 cm\times 1.47\times 10^5 cm}{(2.95\times 10^5 cm)^2} = .747, \tag{5.79}$$

$$\frac{\omega_{pRN1}}{\omega_\phi} = -\frac{(1.47\times 10^5 cm)^2}{2(2.95\times 10^5 cm)^2} = -.125, \tag{5.80}$$

$$\frac{\omega_{pRN2}}{\omega_\phi} = -\frac{3(1.47\times 10^5 cm)^2(1.47\times 10^5 cm)^2}{(2.95\times 10^5 cm)^4} = -.186, \tag{5.81}$$

$$\frac{\omega_{pES}}{\omega_\phi} = \frac{3(1.47\times 10^5 cm)^2(1.47\times 10^5 cm)^2}{(2.95\times 10^5 cm)^6\times 10^{66} cm^{-2}} = 2.14\times 10^{-78}. \tag{5.82}$$

From (5.47,5.9), the pericenter advances are

$$\omega_{pSR} = 0\omega_\phi = 0 rad/s, \tag{5.83}$$

$$\omega_{pGR} = .747\omega_\phi = 1.90\times 10^4 rad/s, \tag{5.84}$$

$$\omega_{pRN1} = -.125\omega_\phi = -3.19\times 10^3 rad/s, \tag{5.85}$$

$$\omega_{pRN2} = -.186\omega_\phi = -4.74\times 10^3 rad/s, \tag{5.86}$$

$$\omega_{pES} = 2.14\times 10^{-78}\omega_\phi = 5.46\times 10^{-74} rad/s. \tag{5.87}$$

Obviously ω_{pES} is too small to measure for either of the test cases, with a fractional difference from the Einstein-Maxwell result of $< 10^{-78}$. However, this result is one more indication that the Λ-renormalized Einstein-Schrödinger theory closely approximates Einstein-Maxwell theory.

5.2 Deflection and time delay of light

Here we calculate the deflection and time delay of light by assuming null geodesics in the Lorentz force equation (4.10), and the charged solution (3.1-3.7). Null geodesics are found by letting $M_2 = Q_2 = 0$ in the classical hydrodynamics Lorentz force equation (4.11) from §4.2. We will find it convenient to rewrite the equations of §4.2 in terms of the impact parameter "b" instead of "L" and "E". The impact parameter is the distance of closest approach of a line drawn from the initial $r \to \infty$ asymptote of the path, and it is where $d\phi/dt = 1/b$ would occur if the path was not bent. From (4.18,4.20) the impact parameter is given by

$$b = L/E. \tag{5.88}$$

Using (4.22) with $b = L/E$ and $Q_2 = M_2 = 0$ gives

$$0 \;=\; \frac{1}{b^2} - \left(\frac{dr}{d\phi}\frac{1}{r^2\check{c}}\right)^2 - \frac{a}{r^2}. \tag{5.89}$$

For these calculations the factor $\check{c}$ from (3.11) dominates over the extra $1/r^6$ term in "a" from (3.10), so for present purposes we will assume that

$$a = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 1 - 2Mw + Q^2w^2, \qquad w = 1/r. \tag{5.90}$$

From (5.89) with $dr/d\phi = 0$, the distance of closest approach R_0 is related to the impact parameter b by

$$\frac{1}{b^2} = \frac{1}{R_0^2}\left(1 - \frac{2M}{R_0} + \frac{Q^2}{R_0^2}\right). \tag{5.91}$$

To find the angular deflection of light we will use the method of [69]. Integrating (5.89) or using (4.23) with $b = L/E$ and $Q_2 = M_2 = 0$ gives

$$\phi \;=\; \int \frac{dr/r^2}{\check{c}\sqrt{1/b^2 - a/r^2}} = \int \frac{-dw}{\check{c}\sqrt{1/b^2 - aw^2}}\,, \qquad w = 1/r. \tag{5.92}$$

We assume that $\delta\phi \approx M[\partial\phi/\partial M]_{M=Q^2=0} + Q^2[\partial\phi/\partial(Q^2)]_{M=Q^2=0}$, where the differentiation of (5.92) and the substitution $M = Q^2 = 0$ is done before the integration to simplify the calculations. Following [69], the differentiation is done with a fixed R_0 from (5.91) instead of a fixed "b". Using (5.91,5.92), and changing sign by convention gives

$$\begin{aligned}
\delta\phi &= M\left[\frac{\partial}{\partial M}\int_0^{1/R_0}\frac{2dw}{\check{c}\sqrt{1/b^2-aw^2}}\right]_{M=Q=0} \\
&\quad + Q^2\left[\frac{\partial}{\partial(Q^2)}\int_0^{1/R_0}\frac{2dw}{\check{c}\sqrt{1/b^2-aw^2}}\right]_{M=Q=0} && (5.93)\\
&= M\int_0^{1/R_0}dw\left[\frac{2(1/R_0^3-w^3)}{\check{c}(1/b^2-aw^2)^{3/2}}\right]_{M=Q=0} \\
&\quad + Q^2\int_0^{1/R_0}dw\left[\frac{(-1/R_0^4+w^4)}{\check{c}(1/b^2-aw^2)^{3/2}}+\frac{2w^4}{\check{c}^3\Lambda_b\sqrt{1/b^2-aw^2}}\right]_{M=Q=0} && (5.94)\\
&= M\int_0^{1/b}dw\left[\frac{2(1/b^3-w^3)}{(1/b^2-w^2)^{3/2}}\right] \\
&\quad + Q^2\int_0^{1/b}dw\left[\frac{(-1/b^4+w^4)}{(1/b^2-w^2)^{3/2}}+\frac{2w^4}{\Lambda_b\sqrt{1/b^2-w^2}}\right] && (5.95)\\
&= 2M\left[\frac{w/b}{\sqrt{1/b^2-w^2}}-\frac{1/b^2}{\sqrt{1/b^2-w^2}}-\sqrt{1/b^2-w^2}\right]_0^{1/b} \\
&\quad + Q^2\left[\frac{w}{2}\sqrt{1/b^2-w^2}-\frac{3\,asin(wb)}{2b^2}\right]_0^{1/b} \\
&\quad + \frac{Q^2}{\Lambda_b}\left[-\frac{w^3}{2}\sqrt{1/b^2-w^2}-\frac{3w}{4b^2}\sqrt{1/b^2-w^2}+\frac{3\,asin(wb)}{4b^4}\right]_0^{1/b}. && (5.96)
\end{aligned}$$

Therefore the angular deflection is

$$\delta\phi = \frac{4M}{b} - \frac{3\pi Q^2}{4b^2} + \frac{3\pi Q^2}{8\Lambda_b b^4}. \quad (5.97)$$

This same result can also be obtained by another method[78]. Using $w = 1/r$ and $w' = \partial w/\partial\phi$ we can write (5.89) as

$$0 = \frac{1}{b^2} - \frac{w'^2}{\check{c}^2} - aw^2. \quad (5.98)$$

Using (3.10,3.11) for a, $\check{c}$ and keeping only terms which fall off as $1/r^4$ or slower we get

$$0 = \frac{\check{c}^2}{b^2} - w'^2 - \check{c}^2(1 - 2Mw + Q^2w^2)w^2 \tag{5.99}$$

$$\approx \frac{1}{b^2} - w'^2 - w^2 + 2Mw^3 - Q^2w^4 - \frac{2Q^2w^4}{\Lambda_b}\left(\frac{1}{b^2} - w^2\right). \tag{5.100}$$

Here the last term is the modification due to our theory. Taking the derivative of this equation gives

$$0 = 2w'\left[-w'' - w + 3Mw^2 - 2Q^2w^3 - \frac{2Q^2w^3}{\Lambda_b}\left(\frac{2}{b^2} - 3w^2\right)\right]. \tag{5.101}$$

Removing the $2w'$ gives

$$w'' + w = 3Mw^2 - 2Q^2w^3 - \frac{2Q^2w^3}{\Lambda_b}\left(\frac{2}{b^2} - 3w^2\right). \tag{5.102}$$

If the terms on the right-hand side were absent, this equation would have the solution $1/r = w = sin(\phi)/b$, which is a straight line along the equatorial plane in spherical coordinates. The terms on the right-hand side can be regarded as perturbations due to the presence of the stationary body, so we seek a solution of the form

$$w = sin(\phi)/b + f(\phi). \tag{5.103}$$

Substituting this and keeping only the first order terms gives

$$f'' + f = 3M\frac{sin^2(\phi)}{b^2} - 2Q^2\frac{sin^3(\phi)}{b^3} - \frac{2Q^2}{\Lambda_b}\frac{sin^3(\phi)}{b^3}\left(\frac{2}{b^2} - \frac{3sin^2(\phi)}{b^2}\right). \tag{5.104}$$

Using the identities $sin^2(\phi) = (1 - cos(2\phi))/2$, $sin^3(\phi) = (3sin(\phi) - sin(3\phi))/4$ and $sin^5(\phi) = (10sin(\phi) - 5sin(3\phi) + sin(5\phi))/16$ gives

$$f'' + f = \frac{3M}{2b^2}(1 - cos(2\phi)) - \frac{Q^2}{2b^3}\left(1 + \frac{2}{\Lambda_b b^2}\right)(3sin(\phi) - sin(3\phi)) \tag{5.105}$$

$$+\frac{3Q^2}{8\Lambda_b b^5}(10sin(\phi) - 5sin(3\phi) + sin(5\phi)). \tag{5.106}$$

Using $(\phi cos(\phi))'' = (cos\phi - \phi sin\phi)' = -2sin(\phi) - \phi cos(\phi)$, it is easy to see that this has the solution

$$f(\phi) = \frac{3M}{2b^2} + \frac{M}{2b^2}cos(2\phi) - \frac{Q^2}{8b^3}\left(-6 + \frac{3}{\Lambda_b b^2}\right)\left(\phi - \frac{\pi}{2}\right)cos(\phi) \quad (5.107)$$

$$-\frac{Q^2}{8b^3}\left(\frac{1}{2} - \frac{7}{8\Lambda_b b^2}\right)sin(3\phi) - \frac{Q^2}{64\Lambda_b b^5}sin(5\phi). \quad (5.108)$$

So the full perturbed solution is

$$w = \frac{sin(\phi)}{b} + \frac{3M}{2b^2} + \frac{M}{2b^2}cos(2\phi) - \frac{3Q^2}{8b^3}\left(2 - \frac{1}{\Lambda_b b^2}\right)\left(\frac{\pi}{2} - \phi\right)cos(\phi) \quad (5.109)$$

$$-\frac{Q^2}{16b^3}\left(1 - \frac{7}{4\Lambda_b b^2}\right)sin(3\phi) - \frac{Q^2}{64\Lambda_b b^5}sin(5\phi). \quad (5.110)$$

At $r = \infty$, $w = 0$ the unperturbed solution requires $\phi = 0$. So if we set $w = 0$ for the perturbed solution, the resulting ϕ will be half of the deflection, and we change sign by convention $\phi = -\delta\phi/2$. Doing this and neglecting higher order terms gives,

$$0 = \frac{sin(-\delta\phi/2)}{b} + \frac{3M}{2b^2} + \frac{Mcos(\delta\phi)}{2b^2} - \frac{3Q^2}{8b^3}\left(2 - \frac{1}{\Lambda_b b^2}\right)\left(\frac{\pi}{2} + \frac{\delta\phi}{2}\right)cos(\delta\phi/2) \quad (5.111)$$

$$-\frac{Q^2}{16b^3}\left(1 - \frac{7}{4\Lambda_b b^2}\right)sin(-3\delta\phi/2) - \frac{Q^2}{64\Lambda_b b^5}sin(-5\delta\phi/2) \quad (5.112)$$

$$\approx -\frac{\delta\phi}{2b} + \frac{3M}{2b^2} + \frac{M}{2b^2} - \frac{3Q^2}{8b^3}\left(2 - \frac{1}{\Lambda_b b^2}\right)\left(\frac{\pi}{2} + \frac{\delta\phi}{2}\right) \quad (5.113)$$

$$+\frac{Q^2}{16b^3}\left(1 - \frac{7}{4\Lambda_b b^2}\right)\frac{3\delta\phi}{2} + \frac{Q^2}{64\Lambda_b b^5}\frac{5\delta\phi}{2} \quad (5.114)$$

$$\approx -\frac{\delta\phi}{2b} + \frac{2M}{b^2} - \frac{3\pi Q^2}{16b^3}\left(2 - \frac{1}{\Lambda_b b^2}\right). \quad (5.115)$$

The resulting angular deflection is

$$\delta\phi = \overset{\delta\phi_{GR}}{\frac{4M}{b}} - \overset{\delta\phi_{RN}}{\frac{3\pi Q^2}{4b^2}} + \overset{\delta\phi_{ES}}{\frac{3\pi Q^2}{8\Lambda_b b^4}}. \quad (5.116)$$

Here the first term $\delta\phi_{GR}$ is the ordinary general relativistic deflection of light, and this result agrees with [69]. The second term $\delta\phi_{RN}$ is from the Reissner-Nordström

Q^2/r^2 term, and this result agrees with [81, 82, 83]. The last term $\delta\phi_{ES}$ is from our theory.

To find the time delay of light we will use a method much like the one used for computing angular deflection, again from [69]. Using (4.25) with $b = L/E$ and $Q_2 = M_2 = 0$ gives the time dependence

$$t = \int \frac{dr}{a\check{c}\sqrt{1-ab^2/r^2}} = \int \frac{rdr}{a\check{c}\sqrt{r^2-ab^2}}. \tag{5.117}$$

We assume that $\delta t \approx M[\partial t/\partial M]_{M=Q^2=0} + Q^2[\partial t/\partial(Q^2)]_{M=Q^2=0}$, where the differentiation of (5.117) and the substitution $M = Q^2 = 0$ is done before the integration to simplify the calculations. Following [69], the differentiation is done with a fixed R_0 from (5.91) instead of a fixed "b". The integration is done from an initial radius R_i to the distance of closest approach R_0, and then from R_0 to the final radius R_f. Using (5.91,5.117) gives the time delay from R_0 to R_f,

$$\begin{aligned}\delta t &= M\left[\frac{\partial}{\partial M}\int_{R_0}^{R_f}\frac{rdr}{a\check{c}\sqrt{r^2-ab^2}}\right]_{M=Q=0} \\ &\quad + Q^2\left[\frac{\partial}{\partial(Q^2)}\int_{R_0}^{R_f}\frac{rdr}{a\check{c}\sqrt{r^2-ab^2}}\right]_{M=Q=0} \end{aligned}\tag{5.118}$$

$$\begin{aligned} &= M\int_{R_0}^{R_f} dr\left[\frac{2}{a^2\check{c}\sqrt{r^2-ab^2}} + \frac{(-b^2+arb^4/R_0^3)}{a\check{c}(r^2-ab^2)^{3/2}}\right]_{M=Q=0} \\ &\quad + Q^2\int_{R_0}^{R_f} dr\left[-\frac{1/r}{a^2\check{c}\sqrt{r^2-ab^2}} + \frac{(b^2/r - arb^4/R_0^4)}{2a\check{c}(r^2-ab^2)^{3/2}} + \frac{1/r^3}{a\Lambda_b\check{c}^3\sqrt{r^2-ab^2}}\right]_{M=Q=0} \end{aligned}\tag{5.119}$$

$$\begin{aligned} &= M\int_{R_0}^{R_f} dr\left[\frac{2}{\sqrt{r^2-R_0^2}} + \frac{(-R_0^2+rR_0)}{(r^2-R_0^2)^{3/2}}\right] \\ &\quad + Q^2\int_{R_0}^{R_f} dr\left[-\frac{3/r}{2\sqrt{r^2-R_0^2}} + \frac{1/r^3}{\Lambda_b\sqrt{r^2-R_0^2}}\right] \end{aligned}\tag{5.120}$$

$$\begin{aligned} &= M\left[2ln(r+\sqrt{r^2-R_0^2}) + \frac{r}{\sqrt{r^2-R_0^2}} - \frac{R_0}{\sqrt{r^2-R_0^2}}\right]_{R_0}^{R_f} \\ &\quad + Q^2\left[-\frac{3\,acos(R_0/r)}{2R_0}\right]_{R_0}^{R_f} \end{aligned}$$

$$+\frac{Q^2}{2\Lambda_b}\left[\frac{\sqrt{r^2-R_0^2}}{r^2R_0^2}+\frac{acos(R_0/r)}{R_0^3}\right]_{R_0}^{R_f} \tag{5.121}$$

$$= M\left[2ln(r+\sqrt{r^2-R_0^2}\,)+\sqrt{\frac{r-R_0}{r+R_0}}\right]_{R_0}^{R_f}$$

$$-\frac{3Q^2}{2}\left[\frac{acos(R_0/r)}{R_0}\right]_{R_0}^{R_f}$$

$$+\frac{Q^2}{2\Lambda_b}\left[\frac{\sqrt{r^2-R_0^2}}{r^2R_0^2}+\frac{acos(R_0/r)}{R_0^3}\right]_{R_0}^{R_f}. \tag{5.122}$$

The variable r is always positive, so to get the time delay from R_i to R_0 we use this same expression but with $R_f \to R_i$. The resulting time delay in geometrized units is

$$\delta t = \delta t_{GR}+\delta t_{RN}+\delta t_{ES} \tag{5.123}$$

$$= M\left[2ln\left(\frac{(R_i+\sqrt{R_i^2-R_0^2}\,)(R_f+\sqrt{R_f^2-R_0^2}\,)}{R_0^2}\right)+\sqrt{\frac{R_i-R_0}{R_i+R_0}}+\sqrt{\frac{R_f-R_0}{R_f+R_0}}\right]$$

$$-\frac{3Q^2}{2}\left[\frac{acos(R_0/R_i)}{R_0}+\frac{acos(R_0/R_f)}{R_0}\right]$$

$$+\frac{Q^2}{2\Lambda_b}\left[\frac{\sqrt{R_i^2-R_0^2}}{R_i^2R_0^2}+\frac{\sqrt{R_f^2-R_0^2}}{R_f^2R_0^2}+\frac{acos(R_0/R_i)}{R_0^3}+\frac{acos(R_0/R_f)}{R_0^3}\right]. \tag{5.124}$$

Here the first line, δt_{GR} is the ordinary general relativistic time delay, and this result agrees with [69]. The second line δt_{RN} is from the Reissner-Nordström Q^2/r^2 term. The last line δt_{ES} is from our theory.

For a first test case we choose $b=R_i/2=R_f/2=R_0=a_0$, the Bohr radius, and $M=M_P$, $Q=Q_P$, for a proton because this case has some approximate physical relevance, and because the Einstein-Schrödinger term will have the greatest effect for small radii. Using $M^{(geom)}=M^{(cgs)}G/c^2$, $Q^{(geom)}=Q^{(cgs)}\sqrt{G/c^4}$ and $esu=cm^{3/2}g^{1/2}/s$

we have

$$M = M_P = 1.67\times 10^{-24}g\left(\frac{6.67\times10^{-8}cm^3/g\cdot s^2}{(3\times10^{10}cm/s)^2}\right) = 1.24\times10^{-52}cm, \quad (5.125)$$

$$Q = |Q_e| = 4.8\times10^{-10}esu\sqrt{\frac{6.67\times10^{-8}cm^3/g\cdot s^2}{(3\times10^{10}cm/s)^4}} = 1.38\times10^{-34}cm, \quad (5.126)$$

$$b = R_i/2 = R_f/2 = R_0 = a_0 = .529\times10^{-8}cm, \quad (5.127)$$

$$\Lambda_b = 10^{66}cm^{-2}. \quad (5.128)$$

For the angular deflections we get

$$\delta\phi_{GR} = \frac{4\times 1.24\times 10^{-52}cm}{.529\times 10^{-8}cm} = 9.36\times 10^{-44}rad, \quad (5.129)$$

$$\delta\phi_{RN} = -\frac{3\pi(1.38\times 10^{-34}cm)^2}{4(.529\times 10^{-8}cm)^2} = -1.60\times 10^{-51}rad, \quad (5.130)$$

$$\delta\phi_{ES} = \frac{3\pi(1.38\times 10^{-34}cm)^2}{8(.529\times 10^{-8}cm)^4\times 10^{66}cm^{-2}} = 2.85\times 10^{-101}rad. \quad (5.131)$$

For the times delays we get

$$\delta t_{GR} = \frac{1.24\times 10^{-52}cm}{3\times 10^{10}cm/s}\left[4ln(2+\sqrt{3})+\frac{2}{\sqrt{3}}\right] = 2.65\times 10^{-62}s, \quad (5.132)$$

$$\delta t_{RN} = -\frac{3(1.38\times 10^{-34}cm)^2}{2\times 3\times 10^{10}cm/s}\left[\frac{2\pi/3}{.529\times 10^{-8}cm}\right] = -3.75\times 10^{-70}s, \quad (5.133)$$

$$\delta t_{ES} = \frac{(1.38\times 10^{-34}cm)^2}{3\times10^{10}cm/s\times2\times10^{66}cm^{-2}}\left[\frac{\sqrt{3}/2+2\pi/3}{(.529\times10^{-8}cm)^3}\right] = 6.31\times10^{-120}s. \quad (5.134)$$

For reference purposes, these results may be compared to the travel time of light across a Bohr radius, $a_0/c = 1.76\times10^{-19}s$.

For a second test case we choose $M = M_\odot$ because this is the smallest black hole we can expect to observe, and the smallest black hole will create the worst-case observable spatial curvature. We choose an extremal black hole with $Q = M$ because this is the worst-case charge which avoids a naked singularity, and we choose $b = R_i = R_f = 2R_0 = 4M$ because this is close to the gravitational radius. Using

$M^{(geom)} = M^{(cgs)}G/c^2$ we have

$$
\begin{aligned}
M &= Q = M_\odot = 1.99\times10^{33}g\left(\frac{6.67\times10^{-8}cm^3/g\cdot s^2}{(3\times10^{10}cm/s)^2}\right) = 1.47\times10^5 cm, \quad (5.135)\\
b &= R_i = R_f = 2R_0 = 4M = 5.90\times10^5 cm, \quad (5.136)\\
\Lambda_b &= 10^{66}cm^{-2}. \quad (5.137)
\end{aligned}
$$

For the angular deflections we get

$$
\begin{aligned}
\delta\phi_{GR} &= \frac{4\times1.47\times10^5 cm}{5.90\times10^5 cm} = 1.0 rad, \quad (5.138)\\
\delta\phi_{RN} &= -\frac{3\pi(1.47\times10^5 cm)^2}{4(5.90\times10^5 cm)^2} = -.147 rad, \quad (5.139)\\
\delta\phi_{ES} &= \frac{3\pi(1.47\times10^5 cm)^2}{8(5.90\times10^5 cm)^4\times10^{66}cm^{-2}} = 2.11\times10^{-79} rad. \quad (5.140)
\end{aligned}
$$

For the time delays we get

$$
\begin{aligned}
\delta t_{GR} &= \frac{1.47\times10^5}{3\times10^{10}cm/s}\left[4ln(2+\sqrt{3})+\frac{2}{\sqrt{3}}\right] = 3.16\times10^{-5}s, \quad (5.141)\\
\delta t_{RN} &= -\frac{3(1.47\times10^5 cm)^2}{2\times3\times10^{10}cm/s}\left[\frac{4\pi/3}{5.90\times10^5 cm}\right] = -7.73\times10^{-6}s, \quad (5.142)\\
\delta t_{ES} &= \frac{(1.47\times10^5 cm)^2}{3\times10^{10}cm/s\times2\times10^{66}cm^{-2}}\left[\frac{4\sqrt{3}+16\pi/3}{(5.90\times10^5 cm)^3}\right] = 4.18\times10^{-83}s. \quad (5.143)
\end{aligned}
$$

For reference purposes, these results may be compared to the travel time of light across the initial radius, $R_i/c = 1.97\times10^{-5}s$.

The contributions $\delta\phi_{ES}$ and δt_{ES} from the Λ-renormalized Einstein-Schrödinger theory are too tiny to measure, with a fractional difference from the Einstein-Maxwell result of $<10^{-57}$. Again this shows how closely the theory matches Einstein-Maxwell theory.

5.3 Shift in Hydrogen atom energy levels

Here we estimate the energy shift of a Hydrogen atom that would result in our theory as compared to Einstein-Maxwell theory. This is an important case to consider because these energy levels can be measured so accurately. It is also significant because it demonstrates that predictions can be done when additional fields are included in the theory. When a spin-1/2 field is added onto our Lagrangian, the theory predicts the ordinary Dirac equation in curved space. We will only consider the effect of the difference between our electric monopole potential (3.9) and the Reissner-Nordström Q/r potential. We will neglect the difference of the metrics, and in fact we will neglect the difference of the metric from that of flat space. Because of this, we do not expect the calculated energy shift to be accurate in an absolute sense. We are only attempting to get an order of magnitude estimate of the energy shift of our charge solution vs. the Reissner-Nordström solution. Using (3.9) the potential energy difference between the two solutions is

$$\Delta V = Q_e \Delta A_0 = \frac{Q_e^2}{\Lambda_b}\left(\frac{M_e}{r^4} - \frac{4Q_e^2}{5r^5}\right). \tag{5.144}$$

Using this result, an estimate of the shift in the energy levels can be calculated using perturbation theory. It is sufficient to treat the problem non-relativistically. The lowest energy level of a Hydrogen atom is spherically symmetric with

$$\psi_0 = \sqrt{1/\pi a_0^3}\, e^{-r/a_0}. \tag{5.145}$$

Unlike the Reissner-Nordström solution, the vector potential of our charged solution is finite at the origin. However, the origin is at $r_0 = \sqrt{Q}(2/\Lambda_b)^{1/4}$ from (3.16) instead of at $r = 0$. Taking this into account gives

$$\Delta E_0 \approx < \psi_0|\Delta V|\psi_0 > \; = \; Q_e^2 \left(\frac{1}{\pi a_0^3}\right) \int_{r_0}^{\infty} e^{-2r/a_0} \left(\frac{M_e}{\Lambda_b r^4} - \frac{4Q_e^2}{5\Lambda_b r^5}\right) 4\pi r^2 dr \tag{5.146}$$

$$- \; Q_e^2 \left(\frac{1}{\pi a_0^3}\right) \int_0^{r_0} e^{-2r/a_0} \left(\frac{1}{r}\right) 4\pi r^2 dr \tag{5.147}$$

$$\approx \; \frac{4Q_e^2}{a_0^3 \Lambda_b} \int_{r_0}^{\infty} \left(\frac{M_e}{r^2} - \frac{4Q_e^2}{5r^3}\right) dr - \frac{4Q_e^2}{a_0^3} \int_0^{r_0} r dr \tag{5.148}$$

$$= \; \frac{4Q_e^2}{a_0^3 \Lambda_b} \left(\frac{M_e}{r_0} - \frac{2Q_e^2}{5r_0^2}\right) - \frac{4Q_e^2 r_0^2}{a_0^3}. \tag{5.149}$$

Using (3.16) and $Q_e = \sqrt{\alpha}\, l_P$ from (2.36), the M_e term is insignificant and we get

$$\Delta E_0 \approx -\frac{4Q_e^2}{a_0^3 \Lambda_b} \frac{2Q_e^2}{5} \frac{\sqrt{\Lambda_b}}{\sqrt{2} Q_e} - \frac{4Q_e^2}{a_0^3} \frac{\sqrt{2} Q_e}{\sqrt{\Lambda_b}} = -\left(\frac{Q_e^2}{2a_0}\right) \frac{48\sqrt{2\alpha}\, l_P}{5a_0^2 \sqrt{\Lambda_b}}. \tag{5.150}$$

The term in the parenthesis is the ground state energy of a Hydrogen atom. With $E_0 = e^2/2a_0 \sim 13.6eV$, $l_P = 1.6 \times 10^{-33} cm$, $\Lambda_b \sim 10^{66} cm^{-2}$, $h \sim 4 \times 10^{-15} eV \cdot s$, and $a_0 = \hbar^2/m_e e^2 \sim 5 \times 10^{-9} cm$ we get

$$\frac{48\sqrt{2\alpha}\, l_P}{5a_0^2 \sqrt{\Lambda_b}} \sim 10^{-50}, \quad \Delta E_0 \sim \frac{e^2}{2a_0} 10^{-50} \sim 10^{-49} eV, \quad \Delta f_0 \sim \frac{\Delta E_0}{h} \sim 10^{-34} Hz. \tag{5.151}$$

This is clearly unmeasurable.

Chapter 6

Application of Newman-Penrose methods

6.1 Newman-Penrose methods applied to the exact field equations

Here we use Newman-Penrose tetrad formalism to derive several results. In particular, we derive an exact solution for $N_{\sigma\mu}$ in terms of $g_{\sigma\mu}$ and $f_{\sigma\mu}$, and an exact solution of the connection equations (2.59), and we confirm the approximate solutions (2.34,2.35) and (2.62,2.63). We also derive the spin coefficients and Weyl tensor components for our charged solution (3.1-3.7), and show that it has Petrov type-D classification. Throughout this section, Latin letters indicate tetrad indices and Greek letters indicate tensor indices, and we assume $n\!=\!4$ and the definitions

$$\hat{f}^{\nu\mu}\!=\!f^{\nu\mu}\sqrt{2}\,i\Lambda_b^{-1/2},\qquad \hat{j}^{\nu}\!=\!j^{\nu}\sqrt{2}\,i\Lambda_b^{-1/2},\qquad \hat{Q}\!=\!Q\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{6.1}$$

Using the definitions (2.4,2.22) we have

$$W^{\sigma\mu} = \frac{\sqrt{-N}}{\sqrt{-g}} N^{\dashv\mu\sigma} = g^{\sigma\mu} + \hat{f}^{\sigma\mu}. \tag{6.2}$$

Let us consider the following theorem which is similar to one in [51]:

Theorem:Assume $W^{\sigma\mu}$ is a real tensor, $\hat{f}^{\sigma\mu} = W^{[\sigma\mu]}$, and $g^{\sigma\mu} = W^{(\sigma\mu)}$ is an invertible metric which can be put into Newman-Penrose tetrad form

$$g_{ab} = g^{ab} = g^{\alpha\beta} e^a{}_\alpha\, e^b{}_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{6.3}$$

$$l^\sigma = e_1{}^\sigma\,,\ n^\sigma = e_2{}^\sigma\,,\ m^\sigma = e_3{}^\sigma\ \ ,\ m^{*\,\sigma} = e_4{}^\sigma, \tag{6.4}$$

$$l_\sigma = e^2{}_\sigma\,,\ n_\sigma = e^1{}_\sigma\,,\ m_\sigma = -e^4{}_\sigma\,,\ m^*_\sigma = -e^3{}_\sigma, \tag{6.5}$$

$$\delta^\sigma_\mu = e_a{}^\sigma e^a{}_\mu \qquad ,\ \delta^a_b = e_b{}^\sigma e^a{}_\sigma, \tag{6.6}$$

$$\mathbf{e} = det(e^a{}_\nu) = \epsilon^{\alpha\beta\sigma\mu} l_\alpha n_\beta m_\sigma m^*_\mu \tag{6.7}$$

$$\mathbf{e}^* = -\mathbf{e}. \tag{6.8}$$

where l_σ and n_σ are real, m_σ and m^*_σ are complex conjugates. Then tetrads $e^a{}_\nu$ may be chosen such that

$$W^{ab} = W^{\alpha\beta} e^a{}_\alpha \, e^b{}_\beta = \begin{pmatrix} 0 & (1+\check{u}) & 0 & 0 \\ (1-\check{u}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+i\grave{u}) \\ 0 & 0 & -(1-i\grave{u}) & 0 \end{pmatrix}, \tag{6.9}$$

$$\hat{f}^{ab} = \begin{pmatrix} 0 & \check{u} & 0 & 0 \\ -\check{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\grave{u} \\ 0 & 0 & i\grave{u} & 0 \end{pmatrix}, \ \hat{f}^a{}_b = \begin{pmatrix} \check{u} & 0 & 0 & 0 \\ 0 & -\check{u} & 0 & 0 \\ 0 & 0 & i\grave{u} & 0 \\ 0 & 0 & 0 & -i\grave{u} \end{pmatrix}, \hat{f}_{ab} = \begin{pmatrix} 0 & -\check{u} & 0 & 0 \\ \check{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\grave{u} \\ 0 & 0 & -i\grave{u} & 0 \end{pmatrix}, \tag{6.10}$$

where $\grave{u}$ and $\check{u}$ are real, except for null fields with $\hat{f}^\sigma{}_\mu \hat{f}^\mu{}_\sigma = det(\hat{f}^\mu{}_\nu) = 0$, in which case tetrads may be chosen such that

$$W^{ab} = W^{\alpha\beta} e^a{}_\alpha \, e^b{}_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -\acute{u} & -\acute{u} \\ 0 & \acute{u} & 0 & -1 \\ 0 & \acute{u} & -1 & 0 \end{pmatrix}, \tag{6.11}$$

$$\hat{f}^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\acute{u} & -\acute{u} \\ 0 & \acute{u} & 0 & 0 \\ 0 & \acute{u} & 0 & 0 \end{pmatrix}, \quad \hat{f}^a{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \acute{u} & \acute{u} \\ \acute{u} & 0 & 0 & 0 \\ \acute{u} & 0 & 0 & 0 \end{pmatrix}, \quad \hat{f}_{ab} = \begin{pmatrix} 0 & 0 & \acute{u} & \acute{u} \\ 0 & 0 & 0 & 0 \\ -\acute{u} & 0 & 0 & 0 \\ -\acute{u} & 0 & 0 & 0 \end{pmatrix}, \tag{6.12}$$

where $\acute{u}$ is real. If $W^{\sigma\mu}$ is instead Hermitian, things are unchanged except that the scalars "$\grave{u}$" (u grave) and "$\check{u}$" (u check) are imaginary instead of real. The above theorem is proven in Appendix T.

One difference from the usual Newman-Penrose formalism is that gauge freedom is restricted so that only type III tetrad transformations can be used. Covariant derivative is done in the usual fashion,

$$T^a{}_{b|c} \;=\; e^a{}_\sigma e_b{}^\mu T^\sigma{}_{\mu;\tau} e_c{}^\tau = T^a{}_{b,c} + \gamma^a{}_{dc} T^d{}_b - \gamma^d{}_{bc} T^a{}_d. \tag{6.13}$$

For the spin coefficients we will follow the conventions of Chandrasekhar[64],

$$\gamma_{abc} = \frac{1}{2}(\lambda_{abc} + \lambda_{cab} - \lambda_{bca}) = e_a{}^\mu e_{b\mu;\sigma} e_c{}^\sigma, \tag{6.14}$$

$$\gamma_{abc} = -\gamma_{bac} \quad , \quad \gamma^a{}_{ac} = 0, \tag{6.15}$$

$$\lambda_{abc} = (e_{b\sigma,\mu} - e_{b\mu,\sigma}) e_a{}^\sigma e_c{}^\mu = e_{b\sigma,\mu}(e_a{}^\sigma e_c{}^\mu - e_a{}^\mu e_c{}^\sigma) = \gamma_{abc} - \gamma_{cba}, \tag{6.16}$$

$$\lambda_{abc} = -\lambda_{cba}, \tag{6.17}$$

$$\rho = \gamma_{314} \quad , \quad \mu = \gamma_{243} \quad , \quad \tau = \gamma_{312} \quad , \quad \pi = \gamma_{241}, \tag{6.18}$$

$$\kappa = \gamma_{311} \quad , \quad \sigma = \gamma_{313} \quad , \quad \lambda = \gamma_{244} \quad , \quad \nu = \gamma_{242}, \tag{6.19}$$

$$\epsilon \;= (\gamma_{211} + \gamma_{341})/2 \quad , \quad \gamma = (\gamma_{212} + \gamma_{342})/2, \tag{6.20}$$

$$\alpha = (\gamma_{214} + \gamma_{344})/2 \quad , \quad \beta = (\gamma_{213} + \gamma_{343})/2. \tag{6.21}$$

With these coefficients and with other tetrad quantities, complex conjugation causes the exchange $3 \rightarrow 4, 4 \rightarrow 3$. As usual we may also define directional derivative operators,

$$D \;=\; e_1{}^\alpha \frac{\partial}{\partial x^\alpha} \quad , \quad \Delta = e_2{}^\alpha \frac{\partial}{\partial x^\alpha} \quad , \quad \delta = e_3{}^\alpha \frac{\partial}{\partial x^\alpha} \quad , \quad \delta^* = e_4{}^\alpha \frac{\partial}{\partial x^\alpha}. \tag{6.22}$$

Substituting (2.65,2.66) into (2.28) gives the Einstein equations and antisymmetric field equations in tetrad form

$$R_{bd} = 8\pi G\left(T_{bd} - \frac{1}{2}g_{bd}T^a_a\right) - \Lambda_b N_{(bd)} - \Lambda_e g_{bd}$$
$$-\Upsilon^a_{(bd)|a} + \Upsilon^a_{a(b|d)} + \Upsilon^c_{(ba)}\Upsilon^a_{(cd)} + \Upsilon^c_{[ba]}\Upsilon^a_{[cd]} - \Upsilon^c_{(bd)}\Upsilon^a_{ca}, \quad (6.23)$$

$$\Lambda_b N_{[bd]} = 2A_{[d|b]}\sqrt{2}\,i\Lambda_b^{1/2} - \Upsilon^a_{[bd]|a} + \Upsilon^c_{(ba)}\Upsilon^a_{[cd]} + \Upsilon^c_{[ba]}\Upsilon^a_{(cd)} - \Upsilon^c_{[bd]}\Upsilon^a_{ca}. \quad (6.24)$$

The usual Ricci identities will be valid if we define Φ_{ab} values in terms of the right-hand side of (6.23),

$$\Phi_{00} = -R_{11}/2,\ \Phi_{22} = -R_{22}/2,\ \Phi_{02} = -R_{33}/2,\ \Phi_{20} = -R_{44}/2, \quad (6.25)$$

$$\Phi_{01} = -R_{13}/2,\ \Phi_{10} = -R_{14}/2,\ \Phi_{12} = -R_{23}/2,\ \Phi_{21} = -R_{24}/2, \quad (6.26)$$

$$\Phi_{11} = -(R_{12} + R_{34})/4,\ \hat{\Lambda} = R/24 = (R_{12} - R_{34})/12. \quad (6.27)$$

First let us consider the case where we do not have $\hat{f}^\sigma{}_\mu \hat{f}^\mu{}_\sigma = det(\hat{f}^\mu{}_\nu) = 0$. It is easily verified from (6.10) that the scalars are given by[51]

$$\grave{u} = \sqrt{\sqrt{\varpi} - \ell/4}\,, \quad (6.28)$$

$$\check{u} = \sqrt{\sqrt{\varpi} + \ell/4}\,, \quad (6.29)$$

where

$$\varpi = (\ell/4)^2 - \hat{f}/g, \quad (6.30)$$

$$\hat{f} = det(\hat{f}_{\mu\nu})\ \ ,\ \ g = det(g_{\mu\nu}), \quad (6.31)$$

$$\hat{f}/g = -\check{u}^2\grave{u}^2, \quad (6.32)$$

$$\ell = \hat{f}^\sigma{}_\mu \hat{f}^\mu{}_\sigma = 2(\check{u}^2 - \grave{u}^2). \quad (6.33)$$

From (6.2), the fundamental tensor of the Einstein-Schrödinger theory is,

$$N^{\dashv ab} = \frac{\sqrt{-g_\diamond}}{\sqrt{-N_\diamond}}\begin{pmatrix} 0 & (1-\breve{u}) & 0 & 0 \\ (1+\breve{u}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-i\grave{u}) \\ 0 & 0 & -(1+i\grave{u}) & 0 \end{pmatrix}, \tag{6.34}$$

$$N_{bc} = \begin{pmatrix} 0 & (1-\breve{u})\breve{c}/\grave{c} & 0 & 0 \\ (1+\breve{u})\breve{c}/\grave{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-i\grave{u})\grave{c}/\breve{c} \\ 0 & 0 & -(1+i\grave{u})\grave{c}/\breve{c} & 0 \end{pmatrix}, \tag{6.35}$$

where

$$\grave{c} = \frac{1}{\sqrt{1+\grave{u}^2}} = \sqrt{1-\grave{s}^2}\,, \tag{6.36}$$

$$\grave{u} = \grave{s}/\grave{c}, \tag{6.37}$$

$$\breve{c} = \frac{1}{\sqrt{1-\breve{u}^2}} = \sqrt{1+\breve{s}^2}\,, \tag{6.38}$$

$$\breve{u} = \breve{s}/\breve{c}, \tag{6.39}$$

$$\sqrt{-N_\diamond} = \sqrt{-det(N_{ab})} = \frac{i}{\breve{c}\grave{c}}, \tag{6.40}$$

$$\sqrt{-g_\diamond} = \sqrt{-det(g_{ab})} = i. \tag{6.41}$$

Note the correspondence of $\grave{s}, \grave{c}, \grave{u}$ and $\breve{s}, \breve{c}, \breve{u}$ to circular and hyperbolic trigonometry functions.

From (6.10,6.13), Ampere's law (2.47) becomes,

$$\frac{4\pi}{c}\hat{j}^c = \hat{f}^{bc}{}_{,b} + \gamma^b_{ab}\hat{f}^{ac} + \gamma^c_{ab}\hat{f}^{ba}, \tag{6.42}$$

$$\frac{4\pi}{c}\hat{j}^2 = \hat{f}^{12}{}_{,1} + \gamma^3_{13}\hat{f}^{12} + \gamma^4_{14}\hat{f}^{12} + \gamma^2_{34}\hat{f}^{43} + \gamma^2_{43}\hat{f}^{34} \tag{6.43}$$

$$= D\breve{u} - \rho^*\breve{u} - \rho\breve{u} - \rho i\grave{u} + \rho^* i\grave{u} \tag{6.44}$$

$$= D\breve{u} - \rho w - \rho^* w^*, \tag{6.45}$$

$$\frac{4\pi}{c}\hat{j}^1 = \hat{f}^{21}{}_{,2} + \gamma^3_{23}\hat{f}^{21} + \gamma^4_{24}\hat{f}^{21} + \gamma^1_{34}\hat{f}^{43} + \gamma^1_{43}\hat{f}^{34} \tag{6.46}$$

$$= -\Delta\breve{u} - \mu\breve{u} - \mu^*\breve{u} + \mu^* i\grave{u} - \mu i\grave{u} \tag{6.47}$$

$$= -\Delta\breve{u} - \mu w - \mu^* w^*, \tag{6.48}$$

$$\frac{4\pi}{c}\hat{j}^4 = \hat{f}^{34}{}_{,3} + \gamma^1_{31}\hat{f}^{34} + \gamma^2_{32}\hat{f}^{34} + \gamma^4_{12}\hat{f}^{21} + \gamma^4_{21}\hat{f}^{12} \tag{6.49}$$

$$= -i\delta\grave{u} - \pi^* i\grave{u} + \tau i\grave{u} + \tau\breve{u} + \pi^*\breve{u} \tag{6.50}$$

$$= -i\delta\grave{u} + \tau w + \pi^* w^*, \tag{6.51}$$

where

$$w = \breve{u} + i\grave{u} \tag{6.52}$$

The connection equations are easier to work with in contravariant form (2.59) than in covariant form (2.55). Multiplying (2.59) by $\sqrt{-N}/\sqrt{-g}$ and using (6.34,6.13,2.61) and (6.40,6.41) gives

$$0 = \; O_b^{cd} \; = \; \frac{\sqrt{-N_\diamond}}{\sqrt{-g_\diamond}}\left(N^{\dashv cd}{}_{,b} + \gamma^d_{ab}N^{\dashv ca} + \gamma^c_{ab}N^{\dashv ad} + \Upsilon^d_{ab}N^{\dashv ca} + \Upsilon^c_{ba}N^{\dashv ad}\right) + \frac{8\pi}{3c}\left(\hat{j}^{[d}\delta^{c]}_b - \frac{1}{2}\hat{j}^a N_{[ab]}N^{\dashv cd}\right), \tag{6.53}$$

$$0 = \; O_b^{11} \; = \; \Upsilon^1_{2b}(1-\check{u}) + \Upsilon^1_{b2}(1+\check{u}), \tag{6.54}$$

$$0 = \; O_b^{22} \; = \; \Upsilon^2_{1b}(1+\check{u}) + \Upsilon^2_{b1}(1-\check{u}), \tag{6.55}$$

$$0 = \; O_b^{33} \; = \; -\Upsilon^3_{4b}(1-i\grave{u}) - \Upsilon^3_{b4}(1+i\grave{u}), \tag{6.56}$$

$$0 = {}^{\pm}O_b^{12} \; = \; \mp\check{u}_{,b} + (1\mp\check{u})\left(-\frac{(\sqrt{-N_\diamond}\,)_{,b}}{\sqrt{-N_\diamond}} + {}^{\pm}\Upsilon^2_{2b} + {}^{\pm}\Upsilon^1_{b1}\right) + \frac{8\pi}{3c}\left(\pm\hat{j}^{[2}\delta^{1]}_b - \frac{1}{2}\hat{j}^a N_{[ab]}\grave{c}\check{c}(1\mp\check{u})\right) \tag{6.57}$$

$$= \; (1\mp\check{u})\left(\mp\check{u}_{,b}\check{c}^2 - \grave{u}\grave{u}_{,b}\grave{c}^2 + {}^{\pm}\Upsilon^2_{2b} + {}^{\pm}\Upsilon^1_{b1}\right) + \frac{8\pi}{3c}\left(\frac{\pm 1}{(1\pm\check{u})}\hat{j}^{[2}\delta^{1]}_b + i\grave{u}\frac{(1\mp\check{u})}{(1+\grave{u}^2)}\hat{j}^{[4}\delta^{3]}_b\right), \tag{6.58}$$

$$0 = {}^{\pm}O_b^{34} \; = \; \pm i\grave{u}_{,b} + (1\mp i\grave{u})\left(\frac{(\sqrt{-N_\diamond}\,)_{,b}}{\sqrt{-N_\diamond}} - {}^{\pm}\Upsilon^4_{4b} - {}^{\pm}\Upsilon^3_{b3}\right) + \frac{8\pi}{3c}\left(\pm\hat{j}^{[4}\delta^{3]}_b + \frac{1}{2}\hat{j}^a N_{[ab]}\grave{c}\check{c}(1\mp i\grave{u})\right) \tag{6.59}$$

$$= \; (1\mp i\grave{u})\left(\pm i\grave{u}_{,b}\grave{c}^2 - \check{u}\check{u}_{,b}\check{c}^2 - {}^{\pm}\Upsilon^4_{4b} - {}^{\pm}\Upsilon^3_{b3}\right) + \frac{8\pi}{3c}\left(\frac{\pm 1}{(1\pm i\grave{u})}\hat{j}^{[4}\delta^{3]}_b + \check{u}\frac{(1\mp i\grave{u})}{(1-\check{u}^2)}\hat{j}^{[2}\delta^{1]}_b\right), \tag{6.60}$$

$$0 = {}^{\pm}O_b^{24} \; = \; \gamma_{31b}(-(1\pm\check{u}) + (1\mp i\grave{u})) + {}^{\pm}\Upsilon^4_{1b}(1\pm\check{u}) - {}^{\pm}\Upsilon^2_{b3}(1\mp i\grave{u}) \pm \frac{8\pi}{3c}\hat{j}^{[4}\delta^{2]}_b \tag{6.61}$$

$$= \; \mp\gamma_{31b}w + {}^{\pm}\Upsilon^4_{1b}(1\pm\check{u}) - {}^{\pm}\Upsilon^2_{b3}(1\mp i\grave{u}) \pm \frac{8\pi}{3c}\hat{j}^{[4}\delta^{2]}_b, \tag{6.62}$$

$$0 = {}^{\pm}O_b^{13} \; = \; \gamma_{24b}((1\mp\check{u}) - (1\pm i\grave{u})) + {}^{\pm}\Upsilon^3_{2b}(1\mp\check{u}) - {}^{\pm}\Upsilon^1_{b4}(1\pm i\grave{u}) \pm \frac{8\pi}{3c}\hat{j}^{[3}\delta^{1]}_b \tag{6.63}$$

$$= \; \mp\gamma_{24b}w + {}^{\pm}\Upsilon^3_{2b}(1\mp\check{u}) - {}^{\pm}\Upsilon^1_{b4}(1\pm i\grave{u}) \pm \frac{8\pi}{3c}\hat{j}^{[3}\delta^{1]}_b, \tag{6.64}$$

To save space in the equations above we are using the notation,

$$ {}^{-}O_b^{dc} \; = \; {}^{+}O_b^{cd} = \; O_b^{cd} \quad , \quad {}^{-}\Upsilon^d_{cb} = {}^{+}\Upsilon^d_{bc} = \Upsilon^d_{bc}. \tag{6.65}$$

The connection equations (6.54-6.64) can be solved by forming linear combinations of them where all of the Υ^a_{bc} terms cancel except for the desired one. The calculations are done in Appendix U. Splitting the result into symmetric and antisymmetric components gives

$$\Upsilon^2_{(12)} = \check{c}^2\check{u}D\check{u} - \frac{4\pi\check{c}^2\check{u}}{3c}\hat{j}^2, \tag{6.66}$$

$$\Upsilon^1_{(12)} = \check{c}^2\check{u}\Delta\check{u} + \frac{4\pi\check{c}^2\check{u}}{3c}\hat{j}^1, \tag{6.67}$$

$$\Upsilon^4_{(34)} = -\grave{c}^2\grave{u}\delta\grave{u} + \frac{4\pi\grave{c}^2 i\grave{u}}{3c}\hat{j}^4, \tag{6.68}$$

$$\Upsilon^1_{(11)} = \grave{u}D\grave{u}\grave{c}^2 - \check{u}D\check{u}\check{c}^2 + \frac{4\pi\check{u}\check{c}^2}{3c}\hat{j}^2, \tag{6.69}$$

$$\Upsilon^2_{(22)} = \grave{u}\Delta\grave{u}\grave{c}^2 - \check{u}\Delta\check{u}\check{c}^2 - \frac{4\pi\check{u}\check{c}^2}{3c}\hat{j}^1, \tag{6.70}$$

$$\Upsilon^3_{(33)} = \grave{u}\delta\grave{u}\grave{c}^2 - \check{u}\delta\check{u}\check{c}^2 - \frac{4\pi i\grave{u}\grave{c}^2}{3c}\hat{j}^4, \tag{6.71}$$

$$\Upsilon^2_{(11)} = \Upsilon^1_{(22)} = \Upsilon^3_{(44)} = 0, \tag{6.72}$$

$$\Upsilon^2_{(23)} = \frac{i\grave{u}}{2}(\delta\check{u}\check{c}^2 - i\delta\grave{u}\grave{c}^2) - \frac{2\pi i\grave{u}\grave{c}^2}{3c}\hat{j}^4, \tag{6.73}$$

$$\Upsilon^1_{(13)} = -\frac{i\grave{u}}{2}(\delta\check{u}\check{c}^2 + i\delta\grave{u}\grave{c}^2) - \frac{2\pi i\grave{u}\grave{c}^2}{3c}\hat{j}^4, \tag{6.74}$$

$$\Upsilon^3_{(13)} = -\frac{\check{u}}{2}(D\check{u}\check{c}^2 + iD\grave{u}\grave{c}^2) + \frac{2\pi\check{u}\check{c}^2}{3c}\hat{j}^2, \tag{6.75}$$

$$\Upsilon^3_{(23)} = -\frac{\check{u}}{2}(\Delta\check{u}\check{c}^2 - i\Delta\grave{u}\grave{c}^2) - \frac{2\pi\check{u}\check{c}^2}{3c}\hat{j}^1, \tag{6.76}$$

$$\Upsilon^4_{(12)} = -\frac{\check{u}\check{c}^2}{2}\left(\delta\check{u}\frac{\check{c}^2}{\grave{c}^2} + \tau w - \pi^* w^*\right), \tag{6.77}$$

$$\Upsilon^2_{(34)} = -\frac{i\grave{u}\grave{c}^2}{2}\left(iD\grave{u}\frac{\grave{c}^2}{\check{c}^2} + \rho w - \rho^* w^*\right), \tag{6.78}$$

$$\Upsilon^1_{(43)} = -\frac{i\grave{u}\grave{c}^2}{2}\left(i\Delta\grave{u}\frac{\grave{c}^2}{\check{c}^2} - \mu w + \mu^* w^*\right), \tag{6.79}$$

$$\Upsilon^2_{(13)} = \frac{\kappa w\check{u}}{\check{z}}\ ,\ \Upsilon^1_{(24)} = -\frac{\nu w\check{u}}{\check{z}}, \tag{6.80}$$

$$\Upsilon^4_{(13)} = \frac{\sigma w i\grave{u}}{\grave{z}}\ ,\ \Upsilon^3_{(24)} = -\frac{\lambda w i\grave{u}}{\grave{z}}, \tag{6.81}$$

$$\Upsilon^4_{(11)} = \frac{\kappa w^2}{\check{z}}\ ,\ \Upsilon^3_{(22)} = -\frac{\nu w^2}{\check{z}}, \tag{6.82}$$

$$\Upsilon^2_{(33)} = \frac{\sigma w^2}{\grave{z}} \quad , \ \Upsilon^1_{(44)} = -\frac{\lambda w^2}{\grave{z}}, \tag{6.83}$$

$$\Upsilon^2_{[12]} = -\check{c}^2 D\check{u} + \frac{4\pi\check{c}^2}{3c}\hat{j}^2, \tag{6.84}$$

$$\Upsilon^1_{[12]} = -\check{c}^2 \Delta\check{u} - \frac{4\pi\check{c}^2}{3c}\hat{j}^1, \tag{6.85}$$

$$\Upsilon^4_{[34]} = -i\grave{c}^2 \delta\grave{u} - \frac{4\pi\grave{c}^2}{3c}\hat{j}^4, \tag{6.86}$$

$$\Upsilon^2_{[23]} = \frac{1}{2}(\delta\check{u}\check{c}^2 - i\delta\grave{u}\grave{c}^2) - \frac{2\pi\grave{c}^2}{3c}\hat{j}^4, \tag{6.87}$$

$$\Upsilon^1_{[13]} = -\frac{1}{2}(\delta\check{u}\check{c}^2 + i\delta\grave{u}\grave{c}^2) - \frac{2\pi\grave{c}^2}{3c}\hat{j}^4, \tag{6.88}$$

$$\Upsilon^3_{[13]} = \frac{1}{2}(D\check{u}\check{c}^2 + iD\grave{u}\grave{c}^2) - \frac{2\pi\check{c}^2}{3c}\hat{j}^2, \tag{6.89}$$

$$\Upsilon^3_{[23]} = -\frac{1}{2}(\Delta\check{u}\check{c}^2 - i\Delta\grave{u}\grave{c}^2) - \frac{2\pi\check{c}^2}{3c}\hat{j}^1, \tag{6.90}$$

$$\Upsilon^4_{[12]} = \frac{\check{c}^2}{2}\left(\delta\check{u}\frac{\check{c}^2}{\grave{c}^2} + \tau w - \pi^* w^*\right), \tag{6.91}$$

$$\Upsilon^2_{[34]} = \frac{\grave{c}^2}{2}\left(iD\grave{u}\frac{\grave{c}^2}{\check{c}^2} + \rho w - \rho^* w^*\right), \tag{6.92}$$

$$\Upsilon^1_{[43]} = -\frac{\grave{c}^2}{2}\left(i\Delta\grave{u}\frac{\grave{c}^2}{\check{c}^2} - \mu w + \mu^* w^*\right), \tag{6.93}$$

$$\Upsilon^2_{[13]} = -\frac{\kappa w}{\check{z}} \quad , \quad \Upsilon^1_{[24]} = -\frac{\nu w}{\check{z}}, \tag{6.94}$$

$$\Upsilon^4_{[13]} = \frac{\sigma w}{\grave{z}} \quad , \quad \Upsilon^3_{[24]} = \frac{\lambda w}{\grave{z}}, \tag{6.95}$$

where

$$\grave{z} = [(1\pm i\grave{u})^2(1\pm\check{u}) + (1\mp i\grave{u})^2(1\mp\check{u})]/2 = 1 + 2i\check{u}\grave{u} - \grave{u}^2, \tag{6.96}$$

$$\check{z} = [(1\pm\check{u})^2(1\pm i\grave{u}) + (1\mp\check{u})^2(1\mp i\grave{u})]/2 = 1 + 2i\check{u}\grave{u} + \check{u}^2. \tag{6.97}$$

As an error check, it is easy to verify that these results agree with (2.57) and (2.8),

$$\Upsilon^a_{(ba)} = \grave{u}\grave{u}_{,b}\grave{c}^2 - \check{u}\check{u}_{,b}\check{c}^2 + \frac{8\pi}{3c}\left(\check{u}\check{c}^2\delta^{[1}_b\hat{j}^{2]} - i\grave{u}\grave{c}^2\delta^{[3}_b\hat{j}^{4]}\right) \tag{6.98}$$

$$= -\frac{(\sqrt{-g_\diamond})_{,b}}{\sqrt{-g_\diamond}} + \frac{(\sqrt{-N_\diamond})_{,b}}{\sqrt{-N_\diamond}} + \frac{4\pi}{3c}\frac{\sqrt{-g_\diamond}}{\sqrt{-N_\diamond}}\hat{j}^a N_{[ab]}, \tag{6.99}$$

$$\Upsilon^a_{[ba]} = 0. \tag{6.100}$$

The tetrad formalism allows the approximation $|\hat{f}^{\nu}{}_{\mu}| \ll 1$ to be stated somewhat more rigorously as $|\grave{u}| \ll 1$, $|\breve{u}| \ll 1$. From (6.28-6.33), a charged particle will have $\breve{u} \approx \hat{Q}/r^2, \grave{u} = 0$. From (2.37) we have $|\breve{u}|^2 \sim 10^{-66}$ for worst-case fields accessible to measurement, so the approximation $|\hat{f}^{\nu}{}_{\mu}| \ll 1$ is quite valid for almost all cases of interest.

Let us consider the tetrad version of the approximate solution of the connection equations (2.62-2.63), which is calculated in Appendix V. This solution differs from the exact solution (6.66-6.95) only by the factors $\grave{c}$,$\breve{c}$,$\grave{z}$,$\breve{z}$. From (6.96,6.97) and

$$\breve{c} \approx 1 + \breve{u}^2/2, \qquad \grave{c} \approx 1 - \grave{u}^2/2, \tag{6.101}$$

these factors will induce terms which are two orders higher in $\grave{u}$ and $\breve{u}$ than the leading order terms. This confirms that the next higher order terms in (2.62-2.63) will be two orders higher in $f^{\mu}{}_{\nu}$ than the leading order terms, and from (2.37) these terms must be $< 10^{-66}$ of the leading order terms.

Now consider the tetrad version of the approximation (2.34,2.35) for $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$. From (6.36,6.38,6.33,6.35,6.3,6.10) we have, to second order in $\grave{u}$ and $\breve{u}$,

$$\breve{c}/\grave{c} \;\approx\; 1 + \breve{u}^2/2 + \grave{u}^2/2 = \;\; 1 + \breve{u}^2 - \ell/4, \tag{6.102}$$

$$-\grave{c}/\breve{c} \;\approx\; -1 + \breve{u}^2/2 + \grave{u}^2/2 = -1 + \grave{u}^2 + \ell/4, \tag{6.103}$$

$$N_{(ab)} = \begin{pmatrix} 0 & \breve{c}/\grave{c} & 0 & 0 \\ \breve{c}/\grave{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\grave{c}/\breve{c} \\ 0 & 0 & -\grave{c}/\breve{c} & 0 \end{pmatrix} \approx g_{ab} + \hat{f}_a{}^c \hat{f}_{cb} - \frac{1}{4} g_{ab}\, \ell, \tag{6.104}$$

$$N_{[ab]} = \begin{pmatrix} 0 & -\breve{u}\breve{c}/\grave{c} & 0 & 0 \\ \breve{u}\breve{c}/\grave{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\grave{u}\grave{c}/\breve{c} \\ 0 & 0 & -i\grave{u}\grave{c}/\breve{c} & 0 \end{pmatrix} \approx \hat{f}_{ab}. \tag{6.105}$$

These results match the order $\hat{f}^2$ approximations (2.34,2.35). The next higher order terms of (6.102,6.103) will be two orders higher in $\grave{u}$ and $\breve{u}$ than the leading order terms. This confirms that the next higher order terms in (2.34,2.35) will be two orders higher in $\hat{f}^{\mu}{}_{\nu}$ than the leading order terms, and from (2.37) these terms must be $<10^{-66}$ of the leading order terms.

Now let us consider the tetrad version of the charged solution (3.1-3.7). The tetrads are similar to those of the Reissner-Nordström solution[64], except for the $\breve{c}$ factors,

$$e_{1\alpha} = l_\alpha = (1, -1/a\breve{c}, 0, 0) \quad , \quad e_1{}^\alpha = l^\alpha = (1/a\breve{c}, 1, 0, 0), \tag{6.106}$$

$$e_{2\alpha} = n_\alpha = \frac{1}{2}(a\breve{c}, 1, 0, 0) \quad , \quad e_2{}^\alpha = n^\alpha = \frac{1}{2}(1, -a\breve{c}, 0, 0), \tag{6.107}$$

$$e_{3\alpha} = m_\alpha = -r\sqrt{\breve{c}/2}\,(0, 0, 1, i\sin\theta), \quad e_3{}^\alpha = m^\alpha = \frac{1}{r\sqrt{2\breve{c}}}(0, 0, 1, i\csc\theta), \tag{6.108}$$

where "a" is defined with (3.4) and from (6.1,6.36-6.39,3.5) we have

$$\grave{u} = 0 \quad , \quad \grave{s} = 0 \quad , \quad \grave{c} = 1, \tag{6.109}$$

$$\breve{u} = \frac{\breve{s}}{\breve{c}} = \frac{\hat{Q}}{\breve{c}\,r^2}, \tag{6.110}$$

$$\breve{s} = \frac{\hat{Q}}{r^2}, \tag{6.111}$$

$$\breve{c} = \frac{1}{\sqrt{1-\breve{u}^2}} = \sqrt{1+\breve{s}^2} = \sqrt{1+\frac{\hat{Q}^2}{r^4}} \tag{6.112}$$

From (6.2,6.106-6.108,6.1,6.40,6.41,6.7), the tetrad solution matches the solution (3.1,3.2) derived previously,

$$
\begin{aligned}
W^{\sigma\mu} &= e^{\sigma}{}_{a} W^{ab} e_{b}{}^{\mu} && (6.113)\\
&= e^{\sigma}{}_{a} \begin{pmatrix} 0 & 1+\check{u} & 0 & 0 \\ 1-\check{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{a\check{c}} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{a\check{c}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{r\sqrt{2\check{c}}} & \frac{i\csc\theta}{r\sqrt{2\check{c}}} \\ 0 & 0 & \frac{1}{r\sqrt{2\check{c}}} & -\frac{i\csc\theta}{r\sqrt{2\check{c}}} \end{pmatrix} && (6.114)\\
&= \begin{pmatrix} \frac{1}{a\check{c}} & \frac{1}{2} & 0 & 0 \\ 1 & -\frac{a\check{c}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{r\sqrt{2\check{c}}} & \frac{1}{r\sqrt{2\check{c}}} \\ 0 & 0 & \frac{i\csc\theta}{r\sqrt{2\check{c}}} & -\frac{i\csc\theta}{r\sqrt{2\check{c}}} \end{pmatrix} \begin{pmatrix} \frac{(1+\check{u})}{2} & -\frac{(1+\check{u})a\check{c}}{2} & 0 & 0 \\ \frac{(1-\check{u})}{a\check{c}} & (1-\check{u}) & 0 & 0 \\ 0 & 0 & -\frac{1}{r\sqrt{2\check{c}}} & \frac{i\csc\theta}{r\sqrt{2\check{c}}} \\ 0 & 0 & -\frac{1}{r\sqrt{2\check{c}}} & -\frac{i\csc\theta}{r\sqrt{2\check{c}}} \end{pmatrix} && (6.115)\\
&= \frac{1}{\check{c}} \begin{pmatrix} 1/a & -\check{s} & 0 & 0 \\ \check{s} & -a\check{c}^2 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}, && (6.116)\\
1/\mathbf{e} &= det(e_{a}{}^{\nu}) = i\csc\theta/\check{c}r^2, && (6.117)\\
\mathbf{e} &= det(e^{a}{}_{\nu}) = -i\check{c}r^2\sin\theta, && (6.118)\\
\sqrt{-N} &= \sqrt{-N_\diamond}\,\mathbf{e} = i\mathbf{e}/\check{c}\grave{c} = r^2\sin\theta, && (6.119)\\
\sqrt{-g} &= \sqrt{-g_\diamond}\,\mathbf{e} = i\mathbf{e} = \check{c}r^2\sin\theta. && (6.120)
\end{aligned}
$$

Let us calculate the spin coefficients and Wely tensor components for our charged solution, so that it may classified. The nonzero tetrad derivatives are,

$$e_{11,1} = -\left(\frac{1}{a\check{c}}\right)' \quad , \quad e_{20,1} = \frac{(a\check{c})'}{2} \quad , \quad e_{33,2} = -r\sqrt{\check{c}/2}\, i\cos\theta\,, \tag{6.121}$$

$$e_{32,1} = -\sqrt{\check{c}/2} - \frac{r\check{c}'}{2\sqrt{2\check{c}}} = \frac{-\check{c}^2+\check{s}^2}{\sqrt{2\check{c}}\,\check{c}} = \frac{-1}{\sqrt{2\check{c}}\,\check{c}} \quad , \quad e_{33,1} = e_{32,1}\, i\sin\theta\,. \tag{6.122}$$

From these and (6.16), the λ_{abc} coefficients are

$$\lambda_{a1b} = e_{11,1}(e_a{}^1 e_b{}^1 - e_a{}^1 e_b{}^1) = 0, \tag{6.123}$$

$$\lambda_{221} = e_{20,1}(e_2{}^0 e_1{}^1 - e_2{}^1 e_1{}^0) = \frac{(a\check{c})'}{2}, \tag{6.124}$$

$$\lambda_{123} = e_{20,1}(e_1{}^0 e_3{}^1 - e_1{}^1 e_3{}^0) = 0, \tag{6.125}$$

$$\lambda_{223} = e_{20,1}(e_2{}^0 e_3{}^1 - e_2{}^1 e_3{}^0) = 0, \tag{6.126}$$

$$\lambda_{324} = e_{20,1}(e_3{}^0 e_4{}^1 - e_3{}^1 e_4{}^0) = 0, \tag{6.127}$$

$$\lambda_{132} = e_{30,1}(e_1{}^0 e_2{}^1 - e_1{}^1 e_2{}^0) = 0, \tag{6.128}$$

$$\lambda_{233} = -e_{32,1}e_2{}^1 e_3{}^2 - e_{33,1}e_2{}^1 e_3{}^3 = 0, \tag{6.129}$$

$$\lambda_{243} = -e_{42,1}e_2{}^1 e_3{}^2 - e_{43,1}e_2{}^1 e_3{}^3 = -2\left(\frac{-1}{\sqrt{2\check{c}}\,\check{c}}\right)\left(\frac{-a\check{c}}{2}\right)\frac{1}{r\sqrt{2\check{c}}} = -\frac{a}{2r\check{c}}, \tag{6.130}$$

$$\lambda_{441} = e_{42,1}e_4{}^2 e_1{}^1 + e_{43,1}e_4{}^3 e_1{}^1 = 0, \tag{6.131}$$

$$\lambda_{431} = e_{32,1}e_4{}^2 e_1{}^1 + e_{33,1}e_4{}^3 e_1{}^1 = 2\left(\frac{-1}{\sqrt{2\check{c}}\,\check{c}}\right)\frac{1}{r\sqrt{2\check{c}}} = -\frac{1}{r\check{c}^2}, \tag{6.132}$$

$$\lambda_{334} = e_{33,2}(e_3{}^3 e_4{}^2 - e_3{}^2 e_4{}^3) = 2(-r\sqrt{\check{c}/2}\, i\cos\theta)\left(\frac{i\csc\theta}{r\sqrt{2\check{c}}}\right)\frac{1}{r\sqrt{2\check{c}}} = \frac{\cot\theta}{r\sqrt{2\check{c}}}\,. \tag{6.133}$$

From (6.14), the spin coefficients are similar to those of the Reissner-Nordström solution[64], except for the $\check{c}$ factors,

$$\begin{aligned}
\rho &= \gamma_{314} = \lambda_{431} = -\frac{1}{r\check{c}^2}, && (6.134)\\
\mu &= \gamma_{243} = \lambda_{243} = -\frac{a}{2r\check{c}}, && (6.135)\\
\beta &= \frac{1}{2}(\gamma_{213}+\gamma_{343}) = \frac{1}{2}\lambda_{334} = \frac{\cot\theta}{2r\sqrt{2\check{c}}}, && (6.136)\\
\alpha &= \frac{1}{2}(\gamma_{214}+\gamma_{344}) = \frac{1}{2}\lambda_{344} = \frac{-\cot\theta}{2r\sqrt{2\check{c}}}, && (6.137)\\
\gamma &= \frac{1}{2}(\gamma_{212}+\gamma_{342}) = \frac{1}{2}\lambda_{221} = \frac{(a\check{c})'}{4}, && (6.138)\\
\epsilon &= \frac{1}{2}(\gamma_{211}+\gamma_{341}) = 0, && (6.139)\\
\tau &= \gamma_{312} = 0, && (6.140)\\
\pi &= \gamma_{241} = 0, && (6.141)\\
\kappa &= \gamma_{311} = 0, && (6.142)\\
\sigma &= \gamma_{313} = 0, && (6.143)\\
\lambda &= \gamma_{244} = 0, && (6.144)\\
\nu &= \gamma_{242} = 0. && (6.145)
\end{aligned}$$

The Weyl tensor components calculated with MAPLE are

$$\begin{aligned}
\Psi_2 &= -\frac{1}{\check{c}}\left(1+\frac{2\hat{Q}^2}{r^4}\right)\left(\frac{m}{r^3}-\frac{\Lambda_e\hat{Q}^2\hat{V}}{4r^4}+\frac{\Lambda_e}{6}-\frac{\Lambda_e\check{c}}{6}\right)+\frac{\Lambda_e\hat{Q}^2}{6r^4}+\frac{\hat{Q}^2}{2\check{c}r^6}, && (6.146)\\
\Psi_0 &= \Psi_1 = \Psi_3 = \Psi_4 = 0. && (6.147)
\end{aligned}$$

The results $\kappa = \sigma = \lambda = \nu = \epsilon = 0$ and $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ prove that the charged solution (3.1,3.7) has the classification of Petrov type-D, the same as the Reissner-Nordström solution.

Next let us find the exact solution for $N_{\sigma\mu}$ in terms of $g_{\sigma\mu}$ and $f_{\sigma\mu}$. Using (6.10,6.35) we have

$$\hat{f}_{ac}\hat{f}^c{}_b = \begin{pmatrix} 0 & \breve{u}^2 & 0 & 0 \\ \breve{u}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \grave{u}^2 \\ 0 & 0 & \grave{u}^2 & 0 \end{pmatrix}, \; \hat{f}_{ac}\hat{f}^c{}_d\hat{f}^d{}_b = \begin{pmatrix} 0 & -\breve{u}^3 & 0 & 0 \\ \breve{u}^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\grave{u}^3 \\ 0 & 0 & i\grave{u}^3 & 0 \end{pmatrix}, \tag{6.148}$$

$$N_{ab} = \breve{c}\grave{c}\begin{pmatrix} 0 & (1-\breve{u})/\grave{c}^2 & 0 & 0 \\ (1+\breve{u})/\grave{c}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-i\grave{u})/\breve{c}^2 \\ 0 & 0 & -(1+i\grave{u})/\breve{c}^2 & 0 \end{pmatrix}, \tag{6.149}$$

$$= \breve{c}\grave{c}\begin{pmatrix} 0 & (1-\breve{u})(1+\grave{u}^2) & 0 & 0 \\ (1+\breve{u})(1+\grave{u}^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-i\grave{u})(1-\breve{u}^2) \\ 0 & 0 & -(1+i\grave{u})(1-\breve{u}^2) & 0 \end{pmatrix}. \tag{6.150}$$

Using (6.36,6.38,6.31,6.33) gives

$$\breve{c}\grave{c} = \frac{1}{\sqrt{(1-\breve{u}^2)(1+\grave{u}^2)}} = \frac{1}{\sqrt{1-(\breve{u}^2-\grave{u}^2)-\breve{u}^2\grave{u}^2}} = \frac{1}{\sqrt{1-\ell/2+\hat{f}/g}}. \tag{6.151}$$

The exact solution for $N_{\mu\nu}$ is then

$$N_{(\mu\nu)} = \breve{c}\grave{c}((1-\ell/2)g_{\mu\nu} + \hat{f}_{\mu\rho}\hat{f}^\rho{}_\nu), \tag{6.152}$$

$$N_{[\mu\nu]} = \breve{c}\grave{c}((1-\ell/2)\hat{f}_{\mu\nu} + \hat{f}_{\mu\rho}\hat{f}^\rho{}_\alpha\hat{f}^\alpha{}_\nu), \tag{6.153}$$

$$N_{\mu\nu} = \breve{c}\grave{c}((1-\ell/2)\delta^\alpha_\mu + \hat{f}_{\mu\rho}\hat{f}^{\rho\alpha})(g_{\alpha\nu} + \hat{f}_{\alpha\nu}). \tag{6.154}$$

These can be written so that they are approximately correct for any dimension,

$$N_{(\mu\nu)} = \frac{(1-\ell/(n-2))g_{\mu\nu}+\hat f_{\mu\rho}\hat f^{\rho}{}_{\nu}}{\sqrt{1-\ell/(n-2)+\hat f/g}}\,, \tag{6.155}$$

$$N_{[\mu\nu]} = \frac{(1-\ell/(n-2))\hat f_{\mu\nu}+\hat f_{\mu\rho}\hat f^{\rho}{}_{\alpha}\hat f^{\alpha}{}_{\nu}}{\sqrt{1-\ell/(n-2)+\hat f/g}}\,, \tag{6.156}$$

$$N_{\mu\nu} = \frac{(1-\ell/(n-2))\delta^{\alpha}_{\mu}+\hat f_{\mu\rho}\hat f^{\rho\alpha}}{\sqrt{1-\ell/(n-2)+\hat f/g}}(g_{\alpha\nu}+\hat f_{\alpha\nu})\,, \tag{6.157}$$

$$N^{\alpha}_{\alpha} = \frac{n-2\ell/(n-2)}{\sqrt{1-\ell/(n-2)+\hat f/g}}\,, \tag{6.158}$$

$$N_{(\mu\nu)}-\frac{1}{2}g_{(\mu\nu)}N^{\alpha}_{\alpha} = \frac{g_{\mu\nu}(1-n/2)+\hat f_{\mu\rho}\hat f^{\rho}{}_{\nu}}{\sqrt{1-\ell/(n-2)+\hat f/g}}\,. \tag{6.159}$$

Now let us consider the null field case where $\hat f^{\sigma}{}_{\mu}\hat f^{\mu}{}_{\sigma}=det(\hat f^{\mu}{}_{\nu})=0$. Using (6.11) we have

$$N^{\dashv ab} = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & \acute u & \acute u\\ 0 & -\acute u & 0 & -1\\ 0 & -\acute u & -1 & 0\end{pmatrix},\quad N_{ba} = \begin{pmatrix} 2\acute u^2 & 1 & \acute u & \acute u\\ 1 & 0 & 0 & 0\\ -\acute u & 0 & 0 & -1\\ -\acute u & 0 & -1 & 0\end{pmatrix}, \tag{6.160}$$

$$\sqrt{-N_{\diamond}} = \sqrt{-det(N_{ab})}=i, \tag{6.161}$$

$$\sqrt{-g_{\diamond}} = \sqrt{-det(g_{ab})}=i. \tag{6.162}$$

In terms of ordinary Newman-Penrose formalism, we are representing null fields with the three complex Maxwell scalars set to $\phi_0=\hat f_{13}=\acute u,\ \phi_1=0,\ \phi_2=0$. With a type III tetrad transformation we have $\phi_0\rightarrow\phi_0 e^{i\theta}/A, \phi_1\rightarrow\phi_1, \phi_2\rightarrow\phi_2 e^{-i\theta}A$, for arbitrary real functions θ and A. Therefore by performing a type III transformation we may always choose $\acute u$ to be a real constant representing the magnitude of the field. This is sometimes helpful because it reduces the number of terms in $\tilde\Gamma^{\alpha}_{\sigma\mu}$, and when $\hat j^{\nu}=0$

it makes Ampere's law just a relationship between spin coefficients.

From (6.12,6.13), Ampere's law (2.47) for null fields is

$$\frac{4\pi}{c}\hat{j}^c = \hat{f}^{bc}{}_{,b} + \gamma^b_{ab}\hat{f}^{ac} + \gamma^c_{ab}\hat{f}^{ba}, \tag{6.163}$$

$$\begin{aligned}\frac{4\pi}{c}\hat{j}^2 &= \hat{f}^{42}{}_{,4} + \gamma^1_{41}\hat{f}^{42} + \gamma^2_{42}\hat{f}^{42} + \gamma^3_{43}\hat{f}^{42} + \gamma^4_{44}\hat{f}^{42} + \gamma^2_{24}\hat{f}^{42} + \gamma^2_{42}\hat{f}^{24}\\ &+ \hat{f}^{32}{}_{,3} + \gamma^1_{31}\hat{f}^{32} + \gamma^2_{32}\hat{f}^{32} + \gamma^3_{33}\hat{f}^{32} + \gamma^4_{34}\hat{f}^{32} + \gamma^2_{23}\hat{f}^{32} + \gamma^2_{32}\hat{f}^{23}\end{aligned} \tag{6.164}$$

$$= \acute{u}_{,4} + \acute{u}_{,3} + (\gamma_{241} - \gamma_{344} + \gamma_{124} + \gamma_{231} - \gamma_{433} + \gamma_{123})\acute{u}, \tag{6.165}$$

$$= \delta^*\acute{u} + \delta\acute{u} + 2Re(\pi - 2\alpha)\acute{u}, \tag{6.166}$$

$$\frac{4\pi}{c}\hat{j}^1 = \gamma^1_{24}\hat{f}^{42} + \gamma^1_{42}\hat{f}^{24} + \gamma^1_{23}\hat{f}^{32} + \gamma^1_{32}\hat{f}^{23} \tag{6.167}$$

$$= (-\gamma_{242} - \gamma_{232})\acute{u} \tag{6.168}$$

$$= -2Re(\nu)\acute{u}, \tag{6.169}$$

$$\begin{aligned}\frac{4\pi}{c}\hat{j}^4 &= \hat{f}^{24}{}_{,2} + \gamma^1_{21}\hat{f}^{24} + \gamma^2_{22}\hat{f}^{24} + \gamma^3_{23}\hat{f}^{24} + \gamma^4_{24}\hat{f}^{24} + \gamma^4_{24}\hat{f}^{42} + \gamma^4_{42}\hat{f}^{24}\\ &+ \gamma^4_{23}\hat{f}^{32} + \gamma^4_{32}\hat{f}^{23}\end{aligned} \tag{6.170}$$

$$= -\acute{u}_{,2} + (-\gamma_{122} + \gamma_{423} + \gamma_{342} - \gamma_{323})\acute{u} \tag{6.171}$$

$$= -\Delta\acute{u} + (2\gamma - \mu + \lambda^*)\acute{u}, \tag{6.172}$$

The connection equations (2.55) can be solved exactly for null fields. Using (D.9,D.7,D.11) and letting $U_{\alpha\sigma\tau} = \check{\Upsilon}^{(2)}_{\alpha\sigma\tau}$ from (D.12), the order $\hat{f}^4$ solution for $\tilde{\Gamma}_{\alpha\nu\mu}$ is given by,

$$\tilde{\Gamma}_{\alpha\nu\mu} = \Gamma_{\alpha\nu\mu} + \check{\Upsilon}_{[\alpha\mu]\tau}\hat{f}^\tau{}_\nu + \check{\Upsilon}_{[\alpha\nu]\tau}\hat{f}^\tau{}_\mu + \check{\Upsilon}_{(\nu\mu)\tau}\hat{f}^\tau{}_\alpha - \frac{1}{2}\Upsilon^\sigma_{\sigma\alpha}g_{\mu\nu} + \Upsilon^\sigma_{\sigma(\mu}g_{\nu)\alpha} + \check{\Upsilon}_{\alpha\nu\mu}, \tag{6.173}$$

where

$$\Upsilon^\sigma_{\sigma\alpha} = \frac{2}{(n-2)}\check{\Upsilon}_{\sigma\tau\alpha}\hat{f}^{\tau\sigma}, \tag{6.174}$$

$$\check{\Upsilon}_{\alpha\nu\mu} = U_{\alpha\nu\mu} + U_{\alpha\sigma\tau}\hat{f}^{\sigma}{}_{\mu}\hat{f}^{\tau}{}_{\nu} + U_{(\mu\sigma)\tau}\hat{f}^{\sigma}{}_{\alpha}\hat{f}^{\tau}{}_{\nu} - U_{(\nu\sigma)\tau}\hat{f}^{\sigma}{}_{\alpha}\hat{f}^{\tau}{}_{\mu} + U_{[\nu\mu]\sigma}\hat{f}^{\sigma}{}_{\tau}\hat{f}^{\tau}{}_{\alpha}$$
$$+\frac{1}{(n-2)}U_{\sigma\tau\alpha}\hat{f}^{\tau\sigma}\hat{f}_{\mu\nu} + \frac{2}{(n-2)}U_{\sigma\rho\tau}\hat{f}^{\rho\sigma}\hat{f}^{\tau}{}_{[\mu}g_{\nu]\alpha}, \quad (6.175)$$
$$U_{\alpha\nu\mu} = \frac{1}{2}(\hat{f}_{\nu\mu;\alpha} + \hat{f}_{\alpha\mu;\nu} - \hat{f}_{\alpha\nu;\mu}) + \frac{8\pi}{(n-1)}\hat{j}_{[\nu}g_{\mu]\alpha}. \quad (6.176)$$

It happens that for the special case of null fields (6.173-6.176) is exact instead of approximate. This can be proven by substituting (6.175,6.176) into (D.11) and using (6.3,6.10) with constant $\acute{u}$. The only properties of U_{amn} needed to prove this are $U_{amn}=-U_{anm}$ and $U_{223}=U_{224}=0$, and these are easy to see from its definition,

$$U_{anm} = \frac{1}{2}(\hat{f}_{nm;a} + \hat{f}_{am;n} - \hat{f}_{an;m}) + \frac{8\pi}{(n-1)}\hat{j}_{[n}g_{m]a}. \quad (6.177)$$
$$= \frac{1}{2}\Big(\hat{f}_{nm,a} - \gamma^{b}{}_{na}\hat{f}_{bm} - \gamma^{b}{}_{ma}\hat{f}_{nb}$$
$$+ \hat{f}_{am,n} - \gamma^{b}{}_{an}\hat{f}_{bm} - \gamma^{b}{}_{mn}\hat{f}_{ab}$$
$$-\hat{f}_{an,m} + \gamma^{b}{}_{am}\hat{f}_{bn} + \gamma^{b}{}_{nm}\hat{f}_{ab}\Big) + \frac{8\pi}{(n-1)}\hat{j}_{[n}g_{m]a}. \quad (6.178)$$

It is unclear whether we can ever have $\hat{j}^{\sigma}\neq 0$ for null fields, but the solution of the connection equations works even for this case.

Finally, from (6.160,6.3,6.12) we see that the solution (2.34,2.35) for $N_{\sigma\mu}$ in terms of $g_{\sigma\mu}$ and $f_{\sigma\mu}$ is exact instead of approximate for null fields,

$$\hat{f}_{ac}\hat{f}^{c}{}_{b} = \begin{pmatrix} 2\acute{u}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{f}^{a}{}_{c}\hat{f}^{c}{}_{b} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\acute{u}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (6.179)$$
$$N_{(ba)} = g_{ba} + \hat{f}_{bc}\hat{f}^{c}{}_{a} - \frac{1}{4}g_{ab}\hat{f}^{a}{}_{c}\hat{f}^{c}{}_{a}, \qquad N_{[ba]} = \hat{f}_{ba}. \quad (6.180)$$

6.2 Newman-Penrose asymptotically flat $\mathcal{O}(1/r^2)$ expansion of the field equations

Here we solve the LRES field equations to $\mathcal{O}(1/r^2)$ in a Newman-Penrose tetrad frame, assuming an asymptotically flat 1/r expansion of the unknowns, and assuming a retarded time coordinate which remains constant on a surface moving along with any radial propagation of radiation. We consider two main cases. For propagation at the speed-of-light with $k=\omega$ we show that LRES theory and Einstein-Maxwell theory are the same. We demonstrate radiation in the form of electromagnetic and gravitational waves, and peeling behavior of the Weyl scalars, and we show that the Proca equation (2.81) has the trivial solution $\theta_\nu=0$ corresponding to Faraday's law. For propagation different than the speed-of-light with $k<\omega$ and $2\Lambda_b=\omega^2-k^2$, the Proca equation could potentially have Proca-wave solutions, and this analysis could determine whether such solutions have positive or negative energy. In fact what we find is that no Proca-wave solutions exist. This work emulates the analysis of Einstein-Maxwell theory in [54, 55] and to a lesser extent in [84]. It is all implemented in a REDUCE symbolic algebra program[63] called LRES_1OR_RETARDED.TXT.

In the following, Latin letters $a, b...h$ indicate tetrad indices, and Greek letters indicate tensor indices. Let us ignore the θ, ϕ coordinates for the moment. Following

[54] we assume that in t, r coordinates the flat-space tetrads are

$$ {}_0e_a{}^{\nu} = \begin{pmatrix} 1/2 & -1/2 \\ 1 & 1 \end{pmatrix}, \quad {}_0e^b{}_{\nu} = \begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix}, \tag{6.181} $$

$$ {}_0e_{a\nu} = \begin{pmatrix} 1/2 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix}. \tag{6.182} $$

We can check that these satisfy the requirements for Newman-Penrose tetrads

$$ {}_0e^b{}_{\nu}\, {}_0e^{\nu}{}_a = \begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1 \\ -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.183} $$

$$ {}_0g_{\mu\nu} = {}_0e_{\mu}{}^{b}\, {}_0e_{b\nu} = \begin{pmatrix} 1 & 1/2 \\ -1 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.184} $$

$$ {}_0g_{ab} = {}_0e_{a\nu}\, {}_0e^{\nu}{}_b = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1 \\ -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.185} $$

The calculations are done using a retarded time coordinate

$$ u = t - kr/\omega \tag{6.186} $$

where $k = (wavenumber)$, $\omega = (frequency)$, $r = (radius)$. The transformation from t, r coordinates to u, r coordinates has the transformation matrix

$$ T = \begin{pmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial r} \\ \frac{\partial r}{\partial t} & \frac{\partial r}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & -k/\omega \\ 0 & 1 \end{pmatrix}, \tag{6.187} $$

$$ T^{-1} = \begin{pmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial r} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & k/\omega \\ 0 & 1 \end{pmatrix}. \tag{6.188} $$

Transforming the flat-space tetrads and metric to u, r coordinates gives

$$ {}_0e^b{}_\nu = \begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & k/\omega \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k/\omega - 1 \\ 1/2 & k/2\omega + 1/2 \end{pmatrix}, \tag{6.189} $$

$$ {}_0e_a{}^\nu = \begin{pmatrix} 1/2 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k/\omega & 1 \end{pmatrix} = \begin{pmatrix} k/2\omega + 1/2 & -1/2 \\ 1 - k/\omega & 1 \end{pmatrix}, \tag{6.190} $$

$$ {}_0g_{\nu\mu} = \begin{pmatrix} 1 & 0 \\ k/\omega & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & k/\omega \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k/\omega & 1 \end{pmatrix} \begin{pmatrix} 1 & k/\omega \\ 0 & -1 \end{pmatrix} \tag{6.191} $$

$$ = \begin{pmatrix} 1 & k/\omega \\ k/\omega & k^2/\omega^2 - 1 \end{pmatrix}, \tag{6.192} $$

$$ {}_0g^{\nu\mu} = \begin{pmatrix} 1 - k^2/\omega^2 & k/\omega \\ k/\omega & -1 \end{pmatrix}. \tag{6.193} $$

We assume that in cartesian coordinates the $1st$ approximation beyond flat-space has a $1/r$ falloff, the $2nd$ approximation has a $1/r^2$ falloff, etcetera. Considering that ${}_0g_{\mu\nu} = diag(1, -1, -r^2, -r^2 sin^2\theta)$ and ${}_0g^{\mu\nu} = diag(1, -1, -1/r^2, -1/r^2 sin^2\theta)$ in spherical coordinates, we can conclude that in spherical coordinates a $1/r$ falloff should look like $(1/r, 1/r, 1, 1)$ for a covariant vector and $(1/r, 1/r, 1/r^2, 1/r^2)$ for a contravariant vector. Following [54] for the θ, ϕ part of the flat-space tetrads, and using the results above, the covariant tetrads are assumed to be of the form

$$ e^b{}_\nu = {}_0e^b{}_\nu + {}_1e^b{}_\nu + {}_2e^b{}_\nu \tag{6.194} $$

where

$$
{}_0e^b{}_\nu = \begin{pmatrix} 1 & k/\omega - 1 & 0 & 0 \\ 1/2 & k/2\omega + 1/2 & 0 & 0 \\ 0 & 0 & -r/\sqrt{2} & -ir\,sin\theta/\sqrt{2} \\ 0 & 0 & -r/\sqrt{2} & ir\,sin\theta/\sqrt{2} \end{pmatrix}, \tag{6.195}
$$

$$
{}_1e^b{}_\nu = \epsilon \begin{pmatrix} a_0/r & a_1/r & a_2 & a_3 \\ b_0/r & b_1/r & b_2 & b_3 \\ c_0/r & c_1/r & c_2 & c_3 \\ d_0/r & d_1/r & d_2 & d_3 \end{pmatrix}, \tag{6.196}
$$

$$
{}_2e^b{}_\nu = \epsilon^2 \begin{pmatrix} A_0/r^2 & A_1/r^2 & A_2/r & A_3/r \\ B_0/r^2 & B_1/r^2 & B_2/r & B_3/r \\ C_0/r^2 & C_1/r^2 & C_2/r & C_3/r \\ D_0/r^2 & D_1/r^2 & D_2/r & D_3/r \end{pmatrix}. \tag{6.197}
$$

and the contravariant tetrads are assumed to be of the form

$$
e_a{}^\nu = {}_0e_a{}^\nu + {}_1e_a{}^\nu + {}_2e_a{}^\nu \tag{6.198}
$$

where

$$
{}_0e_a{}^{\nu} = \begin{pmatrix} k/2\omega + 1/2 & -1/2 & 0 & 0 \\ 1 - k/\omega & 1 & 0 & 0 \\ 0 & 0 & -1/(r\sqrt{2}) & i/(\sqrt{2}r\, sin\theta) \\ 0 & 0 & -1/(r\sqrt{2}) & -i/(\sqrt{2}r\, sin\theta) \end{pmatrix}, \qquad (6.199)
$$

$$
{}_1e_a{}^{\nu} = \epsilon \begin{pmatrix} a^0/r & a^1/r & a^2/r^2 & a^3/r^2 \\ b^0/r & b^1/r & b^2/r^2 & b^3/r^2 \\ c^0/r & c^1/r & c^2/r^2 & c^3/r^2 \\ d^0/r & d^1/r & d^2/r^2 & d^3/r^2 \end{pmatrix}, \qquad (6.200)
$$

$$
{}_2e_a{}^{\nu} = \epsilon^2 \begin{pmatrix} A^0/r^2 & A^1/r^2 & A^2/r^3 & A^3/r^3 \\ B^0/r^2 & B^1/r^2 & B^2/r^3 & B^3/r^3 \\ C^0/r^2 & C^1/r^2 & C^2/r^3 & C^3/r^3 \\ D^0/r^2 & D^1/r^2 & D^2/r^3 & D^3/r^3 \end{pmatrix}. \qquad (6.201)
$$

These tetrads are a generalization of those used in [54], reducing to the same form for speed-of-light propagation with $k = \omega$. The functions d^i, d_i, D^i, D_i are complex conjugates of the functions c^i, c_i, C^i, C_i. The ϵ parameter is included in the program to keep track of the order of terms, but we set $\epsilon = 1$ in any final result. Using a symbolic algebra linear equation solver, the program calculates the 16 coefficients a^i, b^i, c^i, d^i in terms of the coefficients a_i, b_i, c_i, d_i by solving to $\mathcal{O}(\epsilon)$ the set of 16 equations

$$
({}_0e_a{}^{\nu} + {}_1e_a{}^{\nu})({}_0e^b{}_{\nu} + {}_1e^b{}_{\nu}) = \delta^b_a. \qquad (6.202)
$$

Then it calculates the 16 coefficients A^i, B^i, C^i, D^i in terms of the coefficients a_i, b_i, c_i, d_i,

A_i, B_i, C_i, D_i by solving to $\mathcal{O}(\epsilon^2)$ the set of 16 equations

$$e_a{}^\nu e^b{}_\nu = \delta^b_a. \tag{6.203}$$

Since the contravariant antisymmetric field $f^{\mu\nu}$ satisfies Ampere's law (2.47) exactly in both LRES theory and Einstein-Maxwell theory, we require the dual field

$$f^*_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} f^{\alpha\beta}/2 \tag{6.204}$$

to be the curl of a dual potential

$$f^*_{\mu\nu} = A^*_{\nu,\mu} - A^*_{\mu,\nu}. \tag{6.205}$$

This ensures that Ampere's law is satisfied automatically. Using the same considerations as with the tetrads regarding a $1/r$ falloff in spherical coordinates, the dual potential is assumed to be of the form

$$A^*_\nu = {}_1A^*_\nu + {}_2A^*_\nu \tag{6.206}$$

where

$$ {}_1A^*_\nu = \epsilon \left(h_0/r, \quad h_1/r, \quad h_2, \quad h_3 \right), \tag{6.207}$$

$$ {}_2A^*_\nu = \epsilon^2 \left(H_0/r^2, \quad H_1/r^2, \quad H_2/r, \quad H_3/r \right). \tag{6.208}$$

This dual potential is also used to make the system of equations well defined, because A^*_ν contains only 4 unknowns, and our Proca equation contains only 4 equations. Note that [54] does not use either an ordinary potential or a dual potential, but instead uses the 6 components of the electromagnetic field for unknowns. It is unclear why he does this, since he obtains the same result but with more calculations.

Our unknowns are then the dual potential components h_i, H_i and the tetrad components $a_i, b_i, c_i, A_i, B_i, C_i$. Note that the unknowns d_i, D_i are complex conjugates of c_i, C_i, so c_i', c_i'', C_i', C_i'' determine d_i, D_i. All of the unknowns are assumed to depend only on the three coordinates u, θ, ϕ, and not on r. The goal is to calculate the field equations and then solve them for these unknowns. The first step in calculating the field equations is to calculate the λ_{abc} coefficients, spin coefficients and Riemann tensor, which are found from the equations

$$\lambda_{abc} = e_{b\sigma,\mu}({e_a}^{\sigma}{e_c}^{\mu} - {e_c}^{\sigma}{e_a}^{\mu}), \tag{6.209}$$

$$\gamma_{abc} = \frac{1}{2}(\lambda_{abc} + \lambda_{cab} - \lambda_{bca}) = e_{b\mu;\nu}{e_a}^{\mu}{e_c}^{\nu}, \tag{6.210}$$

$$R_{mnpq} = -\gamma_{mnp,q} + \gamma_{mnq,p} - \gamma_{mnr}{\lambda_p}^r{}_q + \gamma_{mrp}{\gamma^r}_{nq} - \gamma_{mrq}{\gamma^r}_{np}. \tag{6.211}$$

The calculation of λ_{abc} to $\mathcal{O}(1/r^2)$ would ordinarily be very time consuming because of the $\mathcal{O}(r)$ components in ${e^a}_{\nu}$. To speed things up we calculate λ_{abc} to $\mathcal{O}(\epsilon^2)$ and then truncate the result to $\mathcal{O}(1/r^2)$. We checked that this gives the same result as doing the calculation the long way.

To calculate the nonsymmetric Ricci tensor $\tilde{R}_{mn}$ we use the method described in Appendix S. To do this calculation we use $\Upsilon^{\alpha}_{\mu\nu}$ from the solution (2.61-2.64) to the connection equations, and

$$f^{ab} = -\varepsilon^{abcd} f^*_{cd}/2 \tag{6.212}$$

from above. The fundamental tensor is calculated in tetrad form using the following relations from (2.23,C.12,C.14),

$$\sqrt{-N_\diamond} = (1-\hat f^{ab}\hat f_{ba}/4)\sqrt{-g_\diamond}\,, \tag{6.213}$$

$$1/\sqrt{-N_\diamond} = (1+\hat f^{ab}\hat f_{ba}/4)/\sqrt{-g_\diamond}\,, \tag{6.214}$$

$$N^{\dashv ab} = (g^{ab}-\hat f^{ab})\sqrt{-g_\diamond}/\sqrt{-N_\diamond}\,, \tag{6.215}$$

$$N_{bc} = (g_{bc}+\hat f_{bc}+\hat f_{bd}\hat f^{d}{}_{c})\sqrt{-N_\diamond}/\sqrt{-g_\diamond}\,, \tag{6.216}$$

where

$$g_{ab}=g^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{6.217}$$

$$\hat f^{ab} = f^{ab}\sqrt{2}\,i\,\Lambda_b^{-1/2}, \tag{6.218}$$

$$\sqrt{-g_\diamond} = \sqrt{-det(g_{ab})} = i, \tag{6.219}$$

$$\sqrt{-N_\diamond} = \sqrt{-det(N_{ab})}. \tag{6.220}$$

Then we calculate the tetrad equivalent of the source-free field equations, which consist of the symmetric part of the Einstein equations (2.31), and the Proca equation derived from (2.33),

$$\tilde{\mathcal{R}}_{(ab)} + \Lambda_b N_{(ab)} + \Lambda_z g_{ab} = 0, \tag{6.221}$$

$$\varepsilon^{abcd}(\tilde{\mathcal{R}}_{[ab|c]} + \Lambda_b N_{[ab|c]}) = 0. \tag{6.222}$$

Great care is taken to ensure that everything is calculated to $\mathcal{O}(1/r^2)$. To get practical computation time and memory usage when multiplying two expressions, it was essential to determine the min and max powers of $1/r$ in each expression, and then to truncate them to the lowest power of $1/r$ required for their product to be accurate to $\mathcal{O}(1/r^2)$. The calculation of the symmetric field equations (6.221) was checked by doing the calculation another way, using the ordinary Ricci tensor $R_{nq} = R^p_{npq}$ from (6.211), together with an expression for $\tilde{\mathcal{R}}_{ab} - R_{ab}$ similar to $\tilde{G}_{ab} - G_{ab}$ from (2.67). The calculation of the Proca equation (6.222) was also checked in a similar manner, using the approximate Proca equation (2.81).

Now let us consider the solution of the field equations for the speed-of-light propagation case where $k = \omega$ and we do not require $2\Lambda_b = \omega^2 - k^2$. The solution is implemented in the subroutine solvekeqw(). Let us call the $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r^2)$ Einstein equations ${}_1E_{ab}$ and ${}_2E_{ab}$ and the Proca equations ${}_1P_a$ and ${}_2P_a$. Looking first at the $\mathcal{O}(1/r)$ field equations we find that

$$ {}_1E_{12} \;\Rightarrow\; \partial^2 a_1/\partial u^2 = 0, \tag{6.223}$$

$$ {}_1E_{13} - {}_1E_{14} \;\Rightarrow\; \partial^2 a_3/\partial u^2 = -\sqrt{2} sin(\theta)\partial^2 c''_1/\partial u^2, \tag{6.224}$$

$$ {}_1E_{13} + {}_1E_{14} \;\Rightarrow\; \partial^2 a_2/\partial u^2 = -\sqrt{2}\partial^2 c'_1/\partial u^2, \tag{6.225}$$

$$ {}_1E_{11} \;\Rightarrow\; \partial^2 c''_3/\partial u^2 = -sin(\theta)\partial^2 c'_2/\partial u^2, \tag{6.226}$$

$$ {}_1P_1 \;\Rightarrow\; \partial h_1/\partial u = 0. \tag{6.227}$$

This solves the field equations to $\mathcal{O}(1/r)$.

The $\mathcal{O}(1/r^2)$ equations impose similar requirements as the $\mathcal{O}(1/r)$ equations, but are more restrictive,

$$ {}_2E_{22} \;\; \Rightarrow \;\; \partial a_1/\partial u = 0, \tag{6.228} $$

$$ {}_2E_{23} - {}_2E_{24} \;\; \Rightarrow \;\; \partial a_3/\partial u = -\sqrt{2}sin(\theta)\partial c_1''/\partial u, \tag{6.229} $$

$$ {}_2E_{23} + {}_2E_{24} \;\; \Rightarrow \;\; \partial a_2/\partial u = -\sqrt{2}\partial c_1'/\partial u. \tag{6.230} $$

Applying these requirements, E_{33} then requires that either $a_1 = 0$ or

$$ \partial^2 c_2'/\partial u^2 = 0 \;\; and \;\; \partial^2 c_3'/\partial u^2 = -sin(\theta)\partial^2 c_2''/\partial u^2. \tag{6.231} $$

Following [54] we will concentrate on the case $a_1 = 0$. Then we find that

$$ {}_2E_{34} \Rightarrow \partial c_3''/\partial u = -sin(\theta)\partial c_2'/\partial u. \tag{6.232} $$

The remaining field equations do not put any constraints on $\mathcal{O}(\epsilon)$ parameters (a_i, b_i, c_i, h_i), but they can instead be solved to get complicated expressions for $\mathcal{O}(\epsilon^2)$ parameters (A_i, B_i, C_i, H_i) in terms of $\mathcal{O}(\epsilon)$ parameters

$$ {}_2P_2 \;\; \Rightarrow \;\; \partial^2 H_1'/\partial u^2, \tag{6.233} $$

$$ {}_2E_{12} \;\; \Rightarrow \;\; \partial^2 A_1/\partial u^2, \tag{6.234} $$

$$ {}_2E_{11} \;\; \Rightarrow \;\; \partial^2 C_3''/\partial u^2, \tag{6.235} $$

$$ {}_2E_{13} - {}_2E_{14} \;\; \Rightarrow \;\; \partial^2 C_1''/\partial u^2, \tag{6.236} $$

$$ {}_2E_{13} + {}_2E_{14} \;\; \Rightarrow \;\; \partial^2 C_1'/\partial u^2. \tag{6.237} $$

Substituting these expressions solves all of the field equations to $\mathcal{O}(1/r^2)$.

Note that the only requirement on the dual potential is that h_1 is a constant, and we are free to set $h_1 = 0$ since h_ν is a potential. The remaining components h_0, h_2, h_3

are all arbitrary functions of u, θ, ϕ, which is to be expected since plane waves in flat space can have any shape and angular pattern. Also, we find that $F_{ab} = f_{ab}$ to $\mathcal{O}(1/r^2)$. The Proca field and the electric and magnetic fields are,

$$\theta_\rho = 0, \tag{6.238}$$

$$E_r = 0, \quad E_\theta = \frac{1}{r sin\,\theta}\frac{\partial h_3}{\partial u}, \quad E_\phi = -\frac{1}{r}\frac{\partial h_2}{\partial u}, \tag{6.239}$$

$$B_r = 0, \quad B_\theta = \frac{1}{r}\frac{\partial h_2}{\partial u}, \qquad B_\phi = \frac{1}{r sin\,\theta}\frac{\partial h_3}{\partial u}. \tag{6.240}$$

We define the "effective" energy-momentum tensor as $8\pi T_{ab} = G_{ab}$ where G_{ab} is the Einstein tensor formed from the symmetric metric. With $T^{\mu\nu} = e_a{}^\mu T^{ab} e_b{}^\nu$ we set $P_0 = T^{00}$, $P_r = T^{01}$, $P_\theta = T^{02} r$, $P_\phi = T^{03} r sin\,\theta$, where the factors r and $r sin\,\theta$ account for basis vector scaling. The resulting energy and power densities are

$$P_0 = P_r = \frac{1}{4\pi r^2 sin^2\theta}\left[\left(\frac{\partial h_2}{\partial u}\right)^2 sin^2\theta + \left(\frac{\partial h_3}{\partial u}\right)^2\right], \quad P_\theta = P_\phi = 0. \tag{6.241}$$

The Ψ_0 Weyl tensor component is $\mathcal{O}(1/r)$, indicating the presence of gravitational radiation,

$${}_1\Psi_0 = \frac{1}{\sqrt{2}\, r sin\,\theta}\left(2 sin\,\theta \frac{\partial^2 c_2'}{\partial u^2} - i\, sin\,\theta \frac{\partial^2 c_2''}{\partial u^2} - i\frac{\partial^2 c_3'}{\partial u^2}\right). \tag{6.242}$$

The functions c_2', c_2'', c_3' are arbitrary functions of u, θ, ϕ. The Weyl tensor component Ψ_1 is $\mathcal{O}(1/r^2)$, and Ψ_2, Ψ_3, Ψ_4 are of higher order in $1/r$, which indicates the start of peeling behavior. Our calculations were only done to $\mathcal{O}(1/r^2)$, so we could not verify the peeling behavior beyond this order. Note that this peeling behavior is opposite to the usual behavior because we have made our tetrads consistent with [54]. As shown in [64], the tetrad transformation $e^1{}_\nu \leftrightarrow e^2{}_\nu$ has no effect on the metric but causes the exchanges $\Psi_4 \leftrightarrow \Psi_0$, $\Psi_3 \leftrightarrow \Psi_1$, and this transformation would make our results conform to the usual convention.

Finally, it happens that our $\mathcal{O}(1/r^2)$ solution for the tetrads and electromagnetic field solves the $\mathcal{O}(1/r^2)$ Einstein-Maxwell field equations. Therefore, from the standpoint of a Newman-Penrose $1/r$ expansion of the field equations, LRES theory is identical to Einstein-Maxwell theory to $\mathcal{O}(1/r^2)$ for speed-of-light propagation.

Now let us consider the solution of the field equations for the case $k < \omega$ where we require $2\Lambda_b = \omega^2 - k^2$, and $2\Lambda_b = mass^2$ of possible Proca radiation. The solution is implemented in the subroutine solvekltw(). Again we call the $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r^2)$ Einstein equations ${}_1E_{ab}$ and ${}_2E_{ab}$ and the Proca equations ${}_1P_a$ and ${}_2P_a$. We will start with the $\mathcal{O}(1/r)$ equations. In the following, the requirements actually involve 2nd derivatives with respect to u but we are integrating them once with zero constant of integration. This has the effect of excluding possible tetrad solutions involving linear functions of u, which is justified by the bad behavior of such solutions as $t \to \infty$.

$$ {}_1E_{33} + {}_1E_{44} + 2\,{}_1E_{34} \Rightarrow \partial c_2'/\partial u = 0, \tag{6.243}$$

$$ {}_1E_{33} + {}_1E_{44} - 2\,{}_1E_{34} \Rightarrow \partial c_3''/\partial u = 0, \tag{6.244}$$

$$ {}_1E_{33} - {}_1E_{44} \quad \Rightarrow \quad \partial c_3'/\partial u = -sin\,\theta \partial c_2''/\partial u, \tag{6.245}$$

$$ {}_1E_{12} \quad \Rightarrow \quad a_1 = (2b_1 - 4\bar{a})(\omega - k)/(\omega + k) \text{ for some } \bar{a}(\theta, \phi), \tag{6.246}$$

$$ {}_1E_{13} + {}_1E_{14} \quad \Rightarrow \quad \partial c_1'/\partial u = [2(w-k)\partial b_2/\partial u - (w+k)\partial a_2/\partial u]/(2\sqrt{2}\,\omega), \tag{6.247}$$

$$ {}_1E_{13} - {}_1E_{14} \quad \Rightarrow \quad \partial c_1''/\partial u = [2(w-k)\partial b_3/\partial u - (w+k)\partial a_3/\partial u]/(2\sqrt{2}\,\omega sin\theta). \tag{6.248}$$

In the following I am not integrating with respect to u because h_ν is a potential,

$$2{}_1P_2 - {}_1P_1 \;\; \text{or} \;\; 2{}_1P_2 + {}_1P_1 \;\; \Rightarrow \;\; \partial^4 h_1/\partial u^4 + \omega^2 \partial^2 h_1/\partial u^2 = 0, \tag{6.249}$$

$$\Rightarrow \;\; h_1 = \bar{h}_1 sin(\omega u + \check{h}_1) + \tilde{h}_1 + \hat{h}_1 u, \tag{6.250}$$

$${}_1P_3 + {}_1P_4 \;\; \Rightarrow \;\; \partial^4 h_2/\partial u^4 + \omega^2 \partial^2 h_2/\partial u^2 = 0, \tag{6.251}$$

$$\Rightarrow \;\; h_2 = \bar{h}_2 sin(\omega u + \check{h}_2) + \tilde{h}_2 + \hat{h}_2 u, \tag{6.252}$$

$${}_1P_3 - {}_1P_4 \;\; \Rightarrow \;\; \partial^4 h_3/\partial u^4 + \omega^2 \partial^2 h_3/\partial u^2 = 0, \tag{6.253}$$

$$\Rightarrow \;\; h_3 = \bar{h}_3 sin(\omega u + \check{h}_3) + \tilde{h}_3 + \hat{h}_3 u. \tag{6.254}$$

Here $\check{h}_1, \check{h}_2, \check{h}_3$ are constants, and $\bar{h}_1, \tilde{h}_1, \hat{h}_1$,$\bar{h}_2, \tilde{h}_2, \hat{h}_2$,$\bar{h}_3, \tilde{h}_3, \hat{h}_3$ are dependent only on θ, ϕ and not on "u". But with the terms linear in u, some components of $f^*_{\mu\nu}$ become

$$f^*_{23} = -\frac{1}{r}\left(\frac{\partial \bar{h}_1}{\partial \theta} sin(\omega u + \check{h}_1) + \frac{\partial \tilde{h}_1}{\partial \theta} + u\frac{\partial \hat{h}_1}{\partial \theta}\right) - \frac{1}{r^2}\left(\frac{\partial H_1}{\partial \theta} + H_2\right), \tag{6.255}$$

$$f^*_{24} = -\frac{1}{r}\left(\frac{\partial \bar{h}_1}{\partial \phi} sin(\omega u + \check{h}_1) + \frac{\partial \tilde{h}_1}{\partial \phi} + u\frac{\partial \hat{h}_1}{\partial \phi}\right) - \frac{1}{r^2}\left(\frac{\partial H_1}{\partial \phi} + H_3\right), \tag{6.256}$$

$$\begin{aligned} f^*_{34} = {} & \frac{\partial \bar{h}_3}{\partial \theta} sin(\omega u + \check{h}_3) + \frac{\partial \tilde{h}_3}{\partial \theta} + u\frac{\partial \hat{h}_3}{\partial \theta} \\ & -\frac{\partial \bar{h}_2}{\partial \phi} sin(\omega u + \check{h}_2) - \frac{\partial \tilde{h}_2}{\partial \phi} - u\frac{\partial \hat{h}_2}{\partial \phi} + \frac{1}{r}\left(\frac{\partial H_3}{\partial \theta} - \frac{\partial H_2}{\partial \phi}\right). \end{aligned} \tag{6.257}$$

To get good behavior as $t \to \infty$ we have

$$f^*_{23} \text{ and } f^*_{24} \;\; \Rightarrow \;\; \hat{h}_1 = \grave{h}_1 = constant, \tag{6.258}$$

$$f^*_{34} \;\; \Rightarrow \;\; \hat{h}_2 = \omega \partial s(\theta,\phi)/\partial \theta + \acute{h}_2(\theta), \quad \hat{h}_3 = \omega \partial s(\theta,\phi)/\partial \phi + \acute{h}_3(\phi), \tag{6.259}$$

where we are using the new variables $s(\theta,\phi), \acute{h}_2(\theta), \acute{h}_3(\phi), \grave{h}_1$. We may also let $\check{h}_3 = 0$ without any loss of generality. This solves the field equations to $\mathcal{O}(1/r)$.

For the $\mathcal{O}(1/r^2)$ equations if we let

$$s(\theta,\phi) = \frac{1}{\omega}\left(f(\theta,\phi) - \int \acute{h}_2(\theta) d\theta - \int \acute{h}_3(\phi) d\phi\right), \tag{6.260}$$

then the following combination of the Proca equations gives

$$0 = \left(\frac{{}_2P_1}{\omega - k} + \frac{2_2P_2}{\omega + k}\right)\frac{\omega r^2 sin^2\theta}{c(\omega^2 - k^2)} \quad (6.261)$$

$$= \frac{\partial^2 f(\theta,\phi)}{\partial\phi^2} + sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial f(\theta,\phi)}{\partial\theta}\right) + \breve{h}_1\omega\, sin^2\theta, \quad (6.262)$$

where $\breve{h}_1 = \grave{h}_1\omega/(\omega^2 - k^2)$. This is solved by assuming

$$f(\theta,\phi) = v(\theta,\phi) + \bar{v}(\theta), \quad (6.263)$$

$$\partial\bar{v}(\theta)/\partial\theta = (\omega\breve{h}_1 cos\,\theta + \acute{h}_1)/sin\,\theta, \quad (6.264)$$

where $\acute{h}_1$ is another constant. Then $v(\theta,\phi)$ must satisfy the generalized Legendre equation for $l = m = 0$

$$\frac{\partial^2 v(\theta,\phi)}{\partial\phi^2} + sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v(\theta,\phi)}{\partial\theta}\right) = 0, \quad (6.265)$$

which has the unique solution

$$v(\theta,\phi) = (Y_{00}\ spherical\ harmonic) = constant. \quad (6.266)$$

Therefore we have

$$\partial f(\theta,\phi)/\partial\phi = 0, \quad (6.267)$$

$$\partial f(\theta,\phi)/\partial\theta = (\omega\breve{h}_1 cos\,\theta + \acute{h}_1)/sin\,\theta. \quad (6.268)$$

Now let us look at another component of $f^*_{\mu\nu}$,

$$f^*_{13} = \bar{h}_2\omega cos(\omega u + \check{h}_2) + \frac{(\breve{h}_1\omega\, cos\,\theta + \acute{h}_1)}{sin\,\theta} + \frac{1}{r}\left(\frac{\partial H_2}{\partial u} - \frac{\partial h_0}{\partial\theta}\right) - \frac{1}{r^2}\frac{\partial H_0}{\partial\theta}. \quad (6.269)$$

To make f^*_{13} finite for $\theta = 0$ or π (the z-axis) we must have $\breve{h}_1 = 0$ and $\acute{h}_1 = 0$. Applying these results and forming a combination of the Einstein equations gives,

$$0 = -\left(\frac{2_2E_{11}}{\omega + k} + \frac{{}_2E_{34}}{\omega - k}\right)\frac{4r^2 sin^2\theta}{c^2(\omega + k)(\omega^2 - k^2)} = \frac{\bar{h}_1^2\omega^2 sin^2\theta}{\omega^2 - k^2} + \bar{h}_2^2 sin^2\theta + \bar{h}_3^2. \quad (6.270)$$

The sum of positive numbers must be positive so this requires $\bar{h}_1 = \bar{h}_2 = \bar{h}_3 = 0$. Therefore h_1, h_2, h_3 have no wavelike component but are just functions of θ and ϕ.

From the above result we can tell that there are not going to be any Proca wave solutions. However let us continue to solve the equations and see what we get. Next we look at some additional combinations of the Proca equation

$$
\begin{aligned}
{}_2P_1 \quad &\Rightarrow \quad \frac{\partial^4 H_1}{\partial u^4} + \omega^2 \frac{\partial^2 H_1}{\partial u^2} + \frac{\partial^3 h_0}{\partial u^3} + \omega^2 \frac{\partial h_0}{\partial u} = 0, &(6.271)\\
{}_2P_3 + {}_2P_4 \quad &\Rightarrow \quad \frac{\partial^4 H_2}{\partial u^4} + \omega^2 \frac{\partial^2 H_2}{\partial u^2} - \frac{\partial^4 h_0}{\partial\theta\partial u^3} - \omega^2 \frac{\partial^2 h_0}{\partial\theta\partial u} = 0, &(6.272)\\
{}_2P_3 - {}_2P_4 \quad &\Rightarrow \quad \frac{\partial^4 H_3}{\partial u^4} + \omega^2 \frac{\partial^2 H_3}{\partial u^2} - \frac{\partial^4 h_0}{\partial\phi\partial u^3} - \omega^2 \frac{\partial^2 h_0}{\partial\phi\partial u} = 0. &(6.273)
\end{aligned}
$$

These have the general solution,

$$
\begin{aligned}
h_0 &= -\partial H_1/\partial u + \bar{h}_0 sin(\omega u + \check{h}_0) + \tilde{h}_0, &(6.274)\\
H_2 &= -\partial H_1/\partial\theta + \bar{H}_2 sin(\omega u + \check{H}_2) + \tilde{H}_2 + \hat{H}_2 u, &(6.275)\\
H_3 &= -\partial H_1/\partial\phi + \bar{H}_3 sin(\omega u + \check{H}_3) + \tilde{H}_3 + \hat{H}_3 u. &(6.276)
\end{aligned}
$$

Here $\check{h}_0, \check{H}_2, \check{H}_3$ are constants, and $\bar{h}_0, \tilde{h}_0, \hat{h}_0, \bar{H}_2, \tilde{H}_2, \hat{H}_2, \bar{H}_3, \tilde{H}_3, \hat{H}_3$ are functions of only θ, ϕ, and not "u". Looking again at $f^*_{\mu\nu}$ gives

$$
\begin{aligned}
f^*_{23} &= -\frac{1}{r}\frac{\partial\tilde{h}_1}{\partial\theta} - \frac{1}{r^2}(\bar{H}_2 sin(\omega u + \check{H}_2) + \tilde{H}_2 + \hat{H}_2 u), &(6.277)\\
f^*_{24} &= -\frac{1}{r}\frac{\partial\tilde{h}_1}{\partial\phi} - \frac{1}{r^2}(\bar{H}_3 sin(\omega u + \check{H}_3) + \tilde{H}_3 + \hat{H}_3 u). &(6.278)
\end{aligned}
$$

To get good asymptotic behavior as $t \to \infty$ requires $\hat{H}_2 = \hat{H}_3 = 0$.

The remaining field equations do not put any constraints on $\mathcal{O}(\epsilon)$ parameters (a_i, b_i, c_i, h_i), but they can instead be solved to get complicated expressions for $\mathcal{O}(\epsilon^2)$ parameters (A_i, B_i, C_i, H_i) in terms of $\mathcal{O}(\epsilon)$ parameters. Substituting these expressions solves all of the field equations to $\mathcal{O}(1/r^2)$.

Again we define the "effective" energy-momentum tensor as $8\pi T_{ab} = G_{ab}$ where G_{ab} is the Einstein tensor formed from the symmetric metric. With $T^{\mu\nu} = e_a{}^{\mu} T^{ab} e_b{}^{\nu}$ we set $P_0 = T^{00}$, $P_r = T^{01}$, $P_\theta = T^{02} r$, $P_\phi = T^{03} r sin\,\theta$, where the factors r and $r sin\,\theta$ account for basis vector scaling. The Proca field, electric and magnetic fields, power densities, and Weyl scalars all indicate no radiation,

$$\theta_\nu = 0 \qquad \text{to } \mathcal{O}(1/r), \tag{6.279}$$

$$E_r = E_\theta = E_\phi = B_r = B_\theta = B_\phi = 0 \qquad \text{to } \mathcal{O}(1/r), \tag{6.280}$$

$$P_0 = P_r = P_\theta = P_\phi = 0 \qquad \text{to } \mathcal{O}(1/r^2), \tag{6.281}$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \qquad \text{to } \mathcal{O}(1/r^2). \tag{6.282}$$

The fact that all of the Weyl scalars vanish indicates that there is no gravitational radiation, which is to be expected because we are requiring propagation at a speed different than the speed-of-light. The lack of any $1/r$ component of θ_ρ or $1/r^2$ component of P_r indicates that there is no propagating Proca radiation. The Proca field θ_a does have $\bar{h}_0 cos(\omega u + \check{h}_0)/r^2$ components, but this does not correspond to propagating radiation and seems of little interest. Also, many of the functions contained in these terms would probably be determined if we were to solve the field equations to a higher order, and it is likely that this would cause these higher order terms to vanish.

So the final result is that LRES theory does not have Proca-wave solutions, given the assumed form of the solution. However, there is some uncertainty as to whether this analysis really rules out Proca-wave solutions, because we may have put too strong of a constraint on the form of the solution. For a wavepacket type of solution, the wavepacket should be expected to spread out as a function of radius, and it seems unlikely that this behavior could be represented by a simple $1/r$ expansion. Also, for a continuous-wave type of solution, the analysis assumes a constant speed of propagation, whereas one might expect the speed of propagation to slow down due to the energy of the wave at smaller radii. There is another issue regarding the propagation speed of Proca waves. The Proca equation is $2\Lambda_b \theta_\rho = -\theta_{\rho\alpha;}{}^{\alpha}$ using a (1,−1,−1,−1) signature. The continuous-wave solution in flat space goes as $sin(\omega t - kx)$, where $k < \omega$ and $2\Lambda_b = \omega^2 - k^2$. From a quantum mechanical viewpoint we have $mass^2 = \hbar^2 2\Lambda_b = (\hbar\omega)^2 - (\hbar k)^2 = energy^2 - momentum^2$, so the "particle" velocity would be $v \approx \text{momentum}/\text{mass} = \hbar k/(\hbar\sqrt{2\Lambda_b}) = k/\sqrt{\omega^2 - k^2} < 1$, which is below the speed of light. This is consistent with the group velocity which is $v_{group} = d\omega/dk = k/\sqrt{2\Lambda_b + k^2} < 1$. However with $k < \omega$ the phase velocity is $v_{phase} = \omega/k = \sqrt{2\Lambda_b + k^2}/k > 1$, so the wavefront (and our retarded coordinate) is travelling at greater than the speed of light! This does not seem right. It makes one wonder if we are not finding a continuous-wave Proca wave solution simply because they are somehow inconsistent with even ordinary general relativity.

Chapter 7

Extension of the Einstein-Schrödinger theory for non-Abelian fields

7.1 The Lagrangian density

Here we generalize LRES theory to non-Abelian fields. The resulting theory incorporates the $U(1)$ and $SU(2)$ gauge terms of the Weinberg-Salam Lagrangian, and when the rest of the Weinberg-Salam Lagrangian is included in a matter term, we get a close approximation to ordinary Einstein-Weinberg-Salam theory. Einstein-Weinberg-Salam theory can be derived from a Palatini Lagrangian density,

$$\begin{aligned}\mathcal{L}(\Gamma^\lambda_{\rho\tau}, g_{\rho\tau}, \mathcal{A}_\nu) \;&=\; -\frac{1}{16\pi}\sqrt{-g}\,[\,g^{\mu\nu}R_{\nu\mu}(\Gamma)+2\Lambda_b\,] \\ &\quad +\frac{1}{32\pi}\sqrt{-g}\;tr(F_{\rho\alpha}g^{\alpha\mu}g^{\rho\nu}F_{\nu\mu})+\mathcal{L}_m(g_{\mu\nu},\mathcal{A}_\nu,\psi,\phi\cdots),\qquad (7.1)\end{aligned}$$

where the electro-weak field tensor is defined as

$$F_{\nu\mu} \;=\; 2\mathcal{A}_{[\mu,\nu]}+\frac{ie}{2\hbar sin\theta_w}\,[\mathcal{A}_\nu,\mathcal{A}_\mu]. \tag{7.2}$$

The Hermitian vector potential $\mathcal{A}_\sigma$ can be decomposed into a real $U(1)$ gauge vector A_σ, and the three real $SU(2)$ gauge vectors b^i_ν,

$$\mathcal{A}_\nu \;=\; IA_\nu+\sigma_i b^i_\nu, \tag{7.3}$$

where the σ_i are the Pauli spin matrices,

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\;\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\;\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\;\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{7.4}$$

$$[\sigma_i,\sigma_j]=2i\epsilon_{ijk}\sigma_k,\quad \sigma_i^\dagger=\sigma_i,\quad tr(\sigma_i)=0,\quad tr(\sigma_i\sigma_j)=2\delta^i_j. \tag{7.5}$$

The $\mathcal{L}_m$ term couples the metric $g_{\mu\nu}$ and vector potential $\mathcal{A}_\mu$ to a spin-1/2 wavefunction ψ, scalar function ϕ, and perhaps the additional fields of the Standard Model. Here and throughout this paper we use geometrized units with $c=G=1$, the symbols $(\,)$ and $[\,]$ around indices indicate symmetrization and antisymmetrization, and $[A,B]=AB-BA$. The constant θ_w is the weak mixing angle and Λ_b is a bare cosmological constant. The factor of 1/2 in (7.2) results because we are including A_ν and b^i_ν in one gauge term $tr(F_{\rho\alpha}g^{\alpha\mu}g^{\rho\nu}F_{\nu\mu})$, and because we are using σ_i instead of the usual $\tau_i=\sigma_i/2$.

The original Einstein-Schrödinger theory allows a nonsymmetric $N_{\mu\nu}$ and $\widehat{\Gamma}^\lambda_{\rho\tau}$ in place of the symmetric $g_{\mu\nu}$ and $\Gamma^\lambda_{\rho\tau}$, and excludes the $tr(F_{\rho\alpha}g^{\alpha\mu}g^{\rho\nu}F_{\nu\mu})$ term. Our "non-Abelian Λ-renormalized Einstein-Schrödinger theory" introduces an additional cosmological term $\mathsf{g}^{1/2d}\Lambda_z$ as in (2.2), and also allows $\widehat{\Gamma}^\rho_{\nu\mu}$ and $N_{\nu\mu}$ to have $d\times d$ matrix components,

$$\mathcal{L}(\widehat{\Gamma}^\alpha_{\rho\tau}, N_{\rho\tau}) = -\frac{1}{16\pi} N^{1/2d}[\, tr(N^{\dashv\mu\nu}\hat{\mathcal{R}}_{\nu\mu}) + d(n-2)\Lambda_b\,] \\ -\frac{1}{16\pi}\,\mathsf{g}^{1/2d} d(n-2)\Lambda_z + \mathcal{L}_m(g_{\mu\nu}, \mathcal{A}_\nu, \psi, \phi\cdots), \tag{7.6}$$

where $\Lambda_b\approx-\Lambda_z$ so that the total Λ matches astronomical measurements[48]

$$\Lambda=\Lambda_b+\Lambda_z\approx 10^{-56}cm^{-2}, \tag{7.7}$$

and the vector potential is defined to be

$$\mathcal{A}_\nu = \widehat{\Gamma}^\sigma_{[\nu\sigma]}/[(n-1)\sqrt{-2\Lambda_b}\,]. \tag{7.8}$$

The $\mathcal{L}_m$ term is not to include a $tr(F_{\rho\alpha}g^{\alpha\mu}g^{\rho\nu}F_{\nu\mu})$ term but may contain the rest of the Weinberg-Salam theory. Matrix indices are assumed to have size $d=2$, and tensor indices are assumed to have dimension n=4, but we will retain "d" and "n" in the equations to show how easily the theory can be generalized. The non-Abelian Ricci tensor is

$$\hat{\mathcal{R}}_{\nu\mu} = \widehat{\Gamma}^\alpha_{\nu\mu,\alpha} - \widehat{\Gamma}^\alpha_{(\alpha(\nu),\mu)} + \frac{1}{2}\widehat{\Gamma}^\sigma_{\nu\mu}\widehat{\Gamma}^\alpha_{(\sigma\alpha)} + \frac{1}{2}\widehat{\Gamma}^\alpha_{(\sigma\alpha)}\widehat{\Gamma}^\sigma_{\nu\mu} - \widehat{\Gamma}^\sigma_{\nu\alpha}\widehat{\Gamma}^\alpha_{\sigma\mu} - \frac{\widehat{\Gamma}^\tau_{[\tau\nu]}\widehat{\Gamma}^\rho_{[\rho\mu]}}{(n-1)}\,. \tag{7.9}$$

For Abelian fields the third and fourth terms are the same, and this tensor reduces to the Abelian version (2.5). This tensor reduces to the ordinary Ricci tensor for $\widehat{\Gamma}^\alpha_{[\nu\mu]} = 0$ and $\widehat{\Gamma}^\alpha_{\alpha[\nu,\mu]} = 0$, as occurs in ordinary general relativity. Let us define the symmetric tensor $\mathsf{g}^{\mu\nu}$ by

$$\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu} = N^{1/2d}N^{\dashv(\mu\nu)}. \tag{7.10}$$

Note that (7.10) defines $\mathsf{g}^{\mu\nu}$ unambiguously because $\mathsf{g} = [det(\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu})]^{2/(n-2)}$. The "physical" metric is denoted with a different symbol $g_{\mu\nu}$, and in this paper we will just be assuming the special case $\mathsf{g}_{\mu\nu} = I g_{\mu\nu}$. The symmetric metric is used for measuring space-time intervals, covariant derivatives, and for raising and lowering indices. If we did not assume $\mathsf{g}_{\mu\nu} = I g_{\mu\nu}$, we would need to choose between several metric definitions which all reduce to the definition (2.4) for Abelian fields,

$$\sqrt{-g}\, g^{\mu\nu} = tr(\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu})/d \quad \text{or} \quad g^{\mu\nu} = tr(\mathsf{g}^{\mu\nu})/d \quad \text{or} \quad g_{\mu\nu} = tr(\mathsf{g}_{\mu\nu})/d, \tag{7.11}$$

and we would also need to choose between

$$\mathsf{g}^{1/2d}\Lambda_z \quad \text{or} \quad \sqrt{-g}\,\Lambda_z \tag{7.12}$$

in the Lagrangian density (7.6). These definitions are all the same with the assumption $\mathsf{g}_{\mu\nu} = I g_{\mu\nu}$, so we will not choose between them here.

The determinants $\mathsf{g} = det(\mathsf{g}_{\nu\mu})$ and $N = det(N_{\nu\mu})$ are defined as usual but where $N_{\nu\mu}$ and $\mathsf{g}_{\nu\mu}$ are taken to be $nd \times nd$ matrices. The inverse of $N_{\nu\mu}$ is defined to be $N^{\dashv\mu k\nu i} = (1/N)\partial N/\partial N_{\nu i\mu k}$ where i,k are matrix indices, or $N^{\dashv\mu\nu} = (1/N)\partial N/\partial N_{\nu\mu}$ using matrix notation. The field $N^{\dashv\mu\nu}$ satisfies the relation $N^{\dashv\mu k\nu i}N_{\nu i\sigma j} = \delta^\mu_\sigma\delta^k_j$, or $N^{\dashv\mu\nu}N_{\nu\sigma} = \delta^\mu_\sigma I$ using matrix notation. Likewise $\mathsf{g}_{\nu\sigma}$ is the inverse of $\mathsf{g}^{\mu\nu}$ such that $\mathsf{g}^{\mu\nu}\mathsf{g}_{\nu\sigma} = \delta^\mu_\sigma I$. Assuming $\bar{N}_{\alpha\tau} = T^\nu_\alpha N_{\nu\mu} T^\mu_\tau$ for some coordinate transformation $T^\nu_\alpha = \partial x^\nu/\partial\bar{x}^\alpha$, the transformed determinant $\bar{N} = det(\bar{N}_{\alpha\tau})$ will contain d times as many T^ν_α factors as it would if $N_{\alpha\tau}$ had no matrix components, so N and g are scalar densities of weight 2d. The factors $N^{1/2d}$ and $\mathsf{g}^{1/2d}$ are used in (7.6) instead of $\sqrt{-N}$ and $\sqrt{-g}$ to make the Lagrangian density a scalar density of weight 1 as required. Note that with an even d, we do not want the factor of -1.

For our theory the electro-weak field tensor $f^{\nu\mu}$ is defined by

$$\mathsf{g}^{1/2d} f^{\nu\mu} = i N^{1/2d} N^{\dashv[\nu\mu]} \Lambda_b^{1/2}/\sqrt{2}. \tag{7.13}$$

Then from (7.10), $\mathsf{g}^{\mu\nu}$ and $f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}$ are parts of a total field,

$$(N/\mathsf{g})^{1/2d}N^{\dashv\nu\mu} = \mathsf{g}^{\mu\nu} + f^{\mu\nu}\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{7.14}$$

We will see that the field equations require $f_{\nu\mu}\approx 2\mathcal{A}_{[\mu,\nu]} + \sqrt{-2\Lambda_b}\,[\mathcal{A}_\nu, \mathcal{A}_\mu]$ to a very high precision. From (7.2,2.3) we see that this agrees with Einstein-Weinberg-Salam theory when

$$-\Lambda_z \approx \Lambda_b = \frac{1}{2}\left(\frac{e}{2\hbar sin\theta_w}\right)^2 = \frac{\alpha}{8l_P^2 sin^2\theta_w} = 1.457\times 10^{63}cm^{-2}, \tag{7.15}$$

where $l_P = \sqrt{G\hbar/c^3} = 1.616\times 10^{-33}cm$, $\alpha = e^2/\hbar c = 1/137$ and $\sin^2\theta_w = .2397$.

It is helpful to decompose $\widehat{\Gamma}^\rho_{\nu\mu}$ into a new connection $\tilde{\Gamma}^\alpha_{\nu\mu}$, and $\mathcal{A}_\nu$ from (7.8),

$$\widehat{\Gamma}^\alpha_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu\mathcal{A}_\nu - \delta^\alpha_\nu\mathcal{A}_\mu)\sqrt{-2\Lambda_b}\,, \tag{7.16}$$

$$\text{where}\quad \tilde{\Gamma}^\alpha_{\nu\mu} = \widehat{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu\widehat{\Gamma}^\sigma_{[\sigma\nu]} - \delta^\alpha_\nu\widehat{\Gamma}^\sigma_{[\sigma\mu]})/(n-1). \tag{7.17}$$

By contracting (7.17) on the right and left we see that $\tilde{\Gamma}^\alpha_{\nu\mu}$ has the symmetry

$$\tilde{\Gamma}^\alpha_{\nu\alpha} = \widehat{\Gamma}^\alpha_{(\nu\alpha)} = \tilde{\Gamma}^\alpha_{\alpha\nu}, \tag{7.18}$$

so it has only n^3-n independent components. Substituting the decomposition (7.16) into (7.9) gives from (R.16),

$$\begin{aligned}\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) = \mathcal{R}_{\nu\mu}(\tilde{\Gamma}) \;&+\; 2\mathcal{A}_{[\nu,\mu]}\sqrt{-2\Lambda_b} + 2\Lambda_b[\mathcal{A}_\nu, \mathcal{A}_\mu]\\ &+\; ([\mathcal{A}_\alpha, \tilde{\Gamma}^\alpha_{\nu\mu}] - [\mathcal{A}_{(\nu}, \tilde{\Gamma}^\alpha_{\mu)\alpha}])\sqrt{-2\Lambda_b}\,.\end{aligned} \tag{7.19}$$

Using (7.19), the Lagrangian density (7.6) can be rewritten in terms of $\tilde{\Gamma}^\alpha_{\nu\mu}$ and $\mathcal{A}_\sigma$

from (7.17,7.8),

$$\mathcal{L} = -\frac{1}{16\pi}N^{1/2d}\Big[tr(N^{\dashv\mu\nu}(\tilde{\mathcal{R}}_{\nu\mu}+2\mathcal{A}_{[\nu,\mu]}\sqrt{-2\Lambda_b}+2\Lambda_b[\mathcal{A}_\nu,\mathcal{A}_\mu]$$
$$+([\mathcal{A}_\alpha,\tilde{\Gamma}^\alpha_{\nu\mu}]-[\mathcal{A}_{(\nu},\tilde{\Gamma}^\alpha_{\mu)\alpha}])\sqrt{-2\Lambda_b}))+d(n-2)\Lambda_b\Big]$$
$$-\frac{1}{16\pi}\,\mathsf{g}^{1/2d}d(n-2)\Lambda_z+\mathcal{L}_m(g_{\mu\nu},A_\sigma,\psi_e,\phi\ldots). \tag{7.20}$$

Here $\tilde{\mathcal{R}}_{\nu\mu}=\mathcal{R}_{\nu\mu}(\tilde{\Gamma})$, and from (7.18) our non-Abelian Ricci tensor (7.9) reduces to

$$\tilde{\mathcal{R}}_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu,\alpha}-\tilde{\Gamma}^\alpha_{\alpha(\nu,\mu)}+\frac{1}{2}\tilde{\Gamma}^\sigma_{\nu\mu}\tilde{\Gamma}^\alpha_{\sigma\alpha}+\frac{1}{2}\tilde{\Gamma}^\alpha_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\nu\mu}-\tilde{\Gamma}^\sigma_{\nu\alpha}\tilde{\Gamma}^\alpha_{\sigma\mu}. \tag{7.21}$$

From (7.16,7.18), $\tilde{\Gamma}^\alpha_{\nu\mu}$ and $\mathcal{A}_\nu$ fully parameterize $\widehat{\Gamma}^\alpha_{\nu\mu}$ and can be treated as independent variables. The fields $N^{1/2d}N^{\dashv(\nu\mu)}$ and $N^{1/2d}N^{\dashv[\nu\mu]}$ (or $\mathsf{g}^{\nu\mu}$ and $f^{\nu\mu}$) fully parameterize $N_{\nu\mu}$ and can also be treated as independent variables. It is simpler to calculate the field equations by setting $\delta\mathcal{L}/\delta\tilde{\Gamma}^\alpha_{\nu\mu}=0$, $\delta\mathcal{L}/\delta\mathcal{A}_\nu=0$, $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv(\mu\nu)})=0$ and $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv[\mu\nu]})=0$ instead of setting $\delta\mathcal{L}/\delta\widehat{\Gamma}^\alpha_{\nu\mu}=0$ and $\delta\mathcal{L}/\delta N_{\nu\mu}=0$, so we will follow this method.

7.2 Invariance properties of the Lagrangian density

Here we show that the Lagrangian density is real (invariant under complex conjugation), and is also invariant under $U(1)$ and $SU(2)$ gauge transformations. The Abelian Lambda-renormalized Einstein-Schrödinger theory comes in two versions, one where $\widehat{\Gamma}^\rho_{\nu\mu}$ and $N_{\nu\mu}$ are real, and one where they are Hermitian. The non-Abelian theory also comes in two versions, one where $\widehat{\Gamma}^\rho_{\nu\mu}$ and $N_{\nu\mu}$ are real, and one where they have

$nd\times nd$ Hermitian symmetry, $\widehat\Gamma^{\alpha\,*}_{\nu i\mu k}=\widehat\Gamma^{\alpha}_{\mu k\nu i}$ and $N^*_{\nu i\mu k}=N_{\mu k\nu i}$, where i,k are matrix indices. Using matrix notation these symmetries become

$$\widehat\Gamma^{\alpha\,*}_{\nu\mu}=\widehat\Gamma^{\alpha\,T}_{\mu\nu},\qquad \tilde\Gamma^{\alpha\,*}_{\nu\mu}=\tilde\Gamma^{\alpha\,T}_{\mu\nu},\qquad N^*_{\nu\mu}=N^T_{\mu\nu},\qquad N^{\dashv\mu\nu\,*}=N^{\dashv\nu\mu\,T},\tag{7.22}$$

where "T" indicates matrix transpose (not transpose over tensor indices). We will assume this Hermitian case because it results from $\Lambda_z<0,\ \Lambda_b>0$ as in (2.12). From (7.22,7.10,7.13,7.8) the physical fields are all composed of $d\times d$ Hermitian matrices,

$$\mathsf{g}^{\nu\mu\,*}=\mathsf{g}^{\nu\mu\,T},\ \mathsf{g}^*_{\nu\mu}=\mathsf{g}^T_{\nu\mu},\ f^{\nu\mu\,*}=f^{\nu\mu\,T},\ f^*_{\nu\mu}=f^T_{\nu\mu},\ \widehat\Gamma^{\alpha\,*}_{(\nu\mu)}=\widehat\Gamma^{\alpha\,T}_{(\nu\mu)},\ \mathcal{A}^*_\nu=\mathcal{A}^T_\nu.\tag{7.23}$$

Hermitian $f_{\nu\mu}$ and $\mathcal{A}_\nu$ are just what we need to approximate Einstein-Weinberg-Salam theory. And of course $\mathsf{g}^{\nu\mu}$ and $\mathsf{g}_{\nu\mu}$ will be Hermitian if we assume the special case where they are multiples of the identity matrix. Writing the symmetries as $N^*_{\nu i\mu k}=N_{\mu k\nu i}$, $\mathsf{g}^*_{\nu i\mu k}=\mathsf{g}_{\nu k\mu i}=\mathsf{g}_{\mu k\nu i}$, and using the result that the determinant of a Hermitian matrix is real, we see that the $nd\times nd$ matrix determinants are real

$$N^*=N,\quad \mathsf{g}^*=\mathsf{g},\quad g^*=g.\tag{7.24}$$

Also, using (7.22) and the identity $M_1^TM_2^T=(M_2M_1)^T$ we can deduce a remarkable property of our non-Abelian Ricci tensor (7.9), which is that it has the same $nd\times nd$ Hermitian symmetry as $\widehat\Gamma^\alpha_{\nu\mu}$ and $N_{\nu\mu}$,

$$\hat{\mathcal{R}}^*_{\nu\mu}=\hat{\mathcal{R}}^T_{\mu\nu}.\tag{7.25}$$

From the properties (7.25,7.22,7.24) and the identities $tr(M_1M_2)=tr(M_2M_1)$, $tr(M^T)=tr(M)$ we see that our Lagrangian density (7.6) or (7.20) is real.

With an $SU(2)$ gauge transformation we assume a transformation matrix U that is special ($det(U)=1$) and unitary ($U^\dagger U=I$). Taking into account (7.3,7.8,7.16), we assume that under an $SU(2)$ gauge transformation the fields transform as follows,

$$B_\nu \rightarrow UB_\nu U^{-1} - \frac{1}{\sqrt{-2\Lambda_b}}\,U_{,\nu}U^{-1}, \tag{7.26}$$

$$\mathcal{A}_\nu \rightarrow U\mathcal{A}_\nu U^{-1} - \frac{1}{\sqrt{-2\Lambda_b}}\,U_{,\nu}U^{-1}, \tag{7.27}$$

$$A_\nu \rightarrow A_\nu, \tag{7.28}$$

$$\widehat{\Gamma}^\alpha_{\nu\mu} \rightarrow U\widehat{\Gamma}^\alpha_{\nu\mu}U^{-1} + 2\delta^\alpha_{[\nu}U_{,\mu]}U^{-1}, \tag{7.29}$$

$$\widehat{\Gamma}^\alpha_{(\nu\mu)} \rightarrow U\widehat{\Gamma}^\alpha_{(\nu\mu)}U^{-1}, \tag{7.30}$$

$$\widehat{\Gamma}^\alpha_{[\alpha\mu]} \rightarrow U\widehat{\Gamma}^\alpha_{[\alpha\mu]}U^{-1} + (n-1)\,U_{,\mu}U^{-1}, \tag{7.31}$$

$$\tilde{\Gamma}^\alpha_{\nu\mu} \rightarrow U\tilde{\Gamma}^\alpha_{\nu\mu}U^{-1}, \tag{7.32}$$

$$N_{\nu\mu} \rightarrow UN_{\nu\mu}U^{-1}, \qquad \mathsf{g}_{\nu\mu} \rightarrow U\mathsf{g}_{\nu\mu}U^{-1}, \qquad f_{\nu\mu} \rightarrow Uf_{\nu\mu}U^{-1}, \tag{7.33}$$

$$N^{\dashv\mu\nu} \rightarrow UN^{\dashv\mu\nu}U^{-1}, \quad \mathsf{g}^{\mu\nu} \rightarrow U\mathsf{g}^{\mu\nu}U^{-1}, \quad f^{\mu\nu} \rightarrow Uf^{\mu\nu}U^{-1}. \tag{7.34}$$

Under a $U(1)$ gauge transformation all of the fields are unchanged except

$$A_\nu \rightarrow A_\nu + \frac{1}{\sqrt{2\Lambda_b}}\,\varphi_{,\nu}, \tag{7.35}$$

$$\mathcal{A}_\nu \rightarrow \mathcal{A}_\nu + \frac{I}{\sqrt{2\Lambda_b}}\,\varphi_{,\nu}, \tag{7.36}$$

$$\widehat{\Gamma}^\alpha_{\nu\mu} \rightarrow \widehat{\Gamma}^\alpha_{\nu\mu} - 2iI\,\delta^\alpha_{[\nu}\varphi_{,\mu]}, \tag{7.37}$$

$$\widehat{\Gamma}^\alpha_{[\alpha\mu]} \rightarrow \widehat{\Gamma}^\alpha_{[\alpha\mu]} - (n-1)iI\,\varphi_{,\mu}. \tag{7.38}$$

Writing the $SU(2)$ gauge transformation (7.33) as

$$N'_{\nu\mu} = \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \end{pmatrix} \begin{pmatrix} N_{00} & N_{01} & N_{02} & N_{03} \\ N_{10} & N_{11} & N_{12} & N_{13} \\ N_{20} & N_{21} & N_{22} & N_{23} \\ N_{30} & N_{31} & N_{32} & N_{33} \end{pmatrix} \begin{pmatrix} U^{-1} & 0 & 0 & 0 \\ 0 & U^{-1} & 0 & 0 \\ 0 & 0 & U^{-1} & 0 \\ 0 & 0 & 0 & U^{-1} \end{pmatrix} \tag{7.39}$$

and using the identity $det(M_1M_2)\!=\!det(M_1)det(M_2)$, we see that the $nd\!\times nd$ matrix determinants are invariant under an $SU(2)$ gauge transformation,

$$N \rightarrow N, \quad \mathsf{g} \rightarrow \mathsf{g}, \quad g \rightarrow g. \tag{7.40}$$

Another remarkable property of our non-Abelian Ricci tensor (7.9) is that it transforms the same as $N_{\nu\mu}$ under an $SU(2)$ gauge transformation (7.29), as in (R.11),

$$\mathcal{R}_{\nu\mu}(U\widehat{\Gamma}^\alpha_{\rho\tau}U^{-1}\!+2\delta^\alpha_{[\rho}U_{,\tau]}U^{-1}) = U\mathcal{R}_{\nu\mu}(\widehat{\Gamma}^\alpha_{\rho\tau})U^{-1} \quad \text{for any matrix } U(x^\sigma). \tag{7.41}$$

The results (7.40,7.41) actually apply for a general matrix U, and do not require that $det(U)=1$ or $U^\dagger U = I$. Using the special case $U = Ie^{-i\varphi}$ in (7.41) we see that our non-Abelian Ricci tensor (7.9) is also invariant under a $U(1)$ gauge transformation,

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}^\alpha_{\rho\tau} - 2iI\,\delta^\alpha_{[\rho}\varphi_{,\tau]}) = \mathcal{R}_{\nu\mu}(\widehat{\Gamma}^\alpha_{\rho\tau}) \qquad \text{for any } \varphi(x^\sigma). \tag{7.42}$$

From (7.41,7.33,7.40,7.42) and the identity $tr(M_1M_2) = tr(M_2M_1)$ we see that our Lagrangian density (7.6) or (7.20) is invariant under both $U(1)$ and $SU(2)$ gauge transformations, thus satisfying an important requirement to approximate Einstein-Weinberg-Salam theory.

One of the motivations for this theory is that the $\Lambda_z = 0$, $\mathcal{L}_m = 0$ version can be derived from a purely affine Lagrangian density as well as a Palatini Lagrangian density, the same as with the Abelian theory in Appendix M. The purely affine Lagrangian density is

$$\mathcal{L}(\widehat{\Gamma}^{\alpha}_{\rho\tau}) = [\,det(N_{\nu\mu})]^{1/2d}, \tag{7.43}$$

where $N_{\nu\mu}$ is simply defined to be

$$N_{\nu\mu} = -\hat{\mathcal{R}}_{\nu\mu}/\Lambda_b. \tag{7.44}$$

Considering that $N^{\dashv\mu\nu} = (1/N)\partial N/\partial N_{\nu\mu}$, we see that setting $\delta\mathcal{L}/\delta\widehat{\Gamma}^{\alpha}_{\rho\tau}=0$ gives the same result obtained from (7.6) with $\Lambda_z=0$, $\mathcal{L}_m=0$,

$$tr[N^{\dashv\mu\nu}\delta\hat{\mathcal{R}}_{\nu\mu}/\delta\widehat{\Gamma}^{\alpha}_{\rho\tau}] = 0. \tag{7.45}$$

Since (7.43) depends only on $\widehat{\Gamma}^{\alpha}_{\rho\tau}$, there are no $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv\mu\nu})=0$ field equations. However, the definition (7.44) exactly matches the $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv\mu\nu})=0$ field equations obtained from (7.6) with $\Lambda_z=0$, $\mathcal{L}_m=0$. Note that there are other definitions of N and g which would make the Lagrangian density (7.6) real and gauge invariant, for example we could have defined $N = tr(\mathrm{det}(\mathrm{N}_{\nu\mu}))$ or $N = \mathrm{Det}(\mathrm{det}(\mathrm{N}_{\nu\mu}))$, where det() is done only over the tensor indices. However, with these definitions the field $N^{\dashv\mu\nu}=(1/N)\partial N/\partial N_{\nu\mu}$ would not be a matrix inverse such that $N^{\dashv\sigma\nu}N_{\nu\mu}=\delta^{\sigma}_{\mu}I$. Calculations would be very unwieldy in a theory where $N^{\dashv\mu\nu}=(1/N)\partial N/\partial N_{\nu\mu}$ appeared in the field equations but was not a genuine inverse of $N_{\nu\mu}$. In addition, it would be impossible to derive the $\Lambda_z = 0$, $\mathcal{L}_m=0$ version of the theory from a purely affine Lagrangian density, thus removing a motivation for the theory. Note that we also cannot use the definition $N=\mathrm{det}(\mathrm{tr}(\mathrm{N}_{\nu\mu}))$ as in [19] because $(-\mathrm{det}(\mathrm{tr}(\mathrm{N}_{\nu\mu})))^{1/2}$ and $(-\mathrm{det}(\mathrm{tr}(\hat{\mathcal{R}}_{\nu\mu})))^{1/2}$ would not depend on the traceless part of the fields.

7.3 The field equations

Let us calculate the field equations for the following special case,

$$\tilde{\Gamma}^{\alpha}_{\nu\mu} = tr(\tilde{\Gamma}^{\alpha}_{\nu\mu})I/d, \quad \mathsf{g}_{\nu\mu} = tr(\mathsf{g}_{\nu\mu})I/d. \tag{7.46}$$

In this case $\mathcal{A}_\nu$ and $N^{1/2d}N^{\dashv[\nu\mu]}$ are the only independent variables in (7.20) which are not just multiples of the identity matrix I. This assumption is both coordinate independent and gauge independent, considering (7.32,7.34). We assume this special case because it gives us Einstein-Weinberg-Salam theory, and because it greatly simplifies the theory. With the assumption (7.46) we also have $\tilde{\mathcal{R}}_{\nu\mu} = tr(\tilde{\mathcal{R}}_{\nu\mu})I/d$, and the term $([\mathcal{A}_\alpha, \tilde{\Gamma}^{\alpha}_{\nu\mu}] - [\mathcal{A}_{(\nu}, \tilde{\Gamma}^{\alpha}_{\mu)\alpha}])\sqrt{-2\Lambda_b}$ vanishes in the Lagrangian density (7.20). And as mentioned initially, with the assumption (7.46) several metric definitions such as (7.11) are the same, so we need not choose one or the other. It would be interesting to investigate the more general theory described by the Lagrangian density (7.6,7.20) without the restriction (7.46). However, it is important to emphasize that any solution of the restricted theory will also be a solution of any of the more general theories which use one of the metric definitions (7.11).

Setting $\delta\mathcal{L}/\delta\mathcal{A}_\tau = 0$ and using the definition (7.13) of $f^{\nu\mu}$ gives the ordinary Weinberg-Salam equivalent of Ampere's law,

$$(\mathsf{g}^{1/2d} f^{\omega\tau})_{,\omega} - \sqrt{-2\Lambda_b}\, \mathsf{g}^{1/2d}[f^{\omega\tau}, \mathcal{A}_\omega] = 4\pi \mathsf{g}^{1/2d} j^\tau, \tag{7.47}$$

where the source current j^τ is defined by

$$j^\tau = \frac{-1}{\mathsf{g}^{1/2d}} \frac{\delta\mathcal{L}_m}{\delta\mathcal{A}_\tau}\,. \tag{7.48}$$

Setting $\delta\mathcal{L}/\delta\tilde\Gamma^\beta_{\tau\rho}=0$ using a Lagrange multiplier term $tr[\Omega^\rho\tilde\Gamma^\alpha_{[\alpha\rho]}]$ to enforce the symmetry (7.18), and using the result $tr[(\mathsf{g}^{1/2d}f^{\omega\tau})_{,\omega}]=4\pi\mathsf{g}^{1/2d}tr[j^\tau]$ derived from (7.47,7.3,7.5) gives the connection equations,

$$tr[(N^{1/2d}N^{\dashv\rho\tau})_{,\beta}+\tilde\Gamma^\tau_{\sigma\beta}N^{1/2d}N^{\dashv\rho\sigma}+\tilde\Gamma^\rho_{\beta\sigma}N^{1/2d}N^{\dashv\sigma\tau}-\tilde\Gamma^\alpha_{\beta\alpha}N^{1/2d}N^{\dashv\rho\tau}]$$
$$=\frac{8\pi\sqrt{2}\,i}{(n-1)\Lambda_b^{1/2}}\,\mathsf{g}^{1/2d}tr[j^{[\rho}]\delta^{\tau]}_\beta. \qquad (7.49)$$

Setting $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv(\mu\nu)})=0$ using the identities $N=[det(N^{1/2d}N^{\dashv\mu\nu})]^{2/(n-2)}$ and $\mathsf{g}=[det(N^{1/2d}N^{\dashv(\mu\nu)})]^{2/(n-2)}$ gives our equivalent of the Einstein equations,

$$tr[\tilde{\mathcal{R}}_{(\nu\mu)}+\Lambda_bN_{(\nu\mu)}+\Lambda_z\mathsf{g}_{\nu\mu}]=8\pi tr[S_{\nu\mu}], \qquad (7.50)$$

where $S_{\nu\mu}$ is defined by

$$S_{\nu\mu}\ \equiv\ 2\frac{\delta\mathcal{L}_m}{\delta(N^{1/2d}N^{(\mu\nu)})}=2\frac{\delta\mathcal{L}_m}{\delta(\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu})}. \qquad (7.51)$$

Setting $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv[\mu\nu]})=0$ using the identities $N=[det(N^{1/2d}N^{\dashv\mu\nu})]^{2/(n-2)}$ and $\mathsf{g}=[det(N^{1/2d}N^{\dashv(\mu\nu)})]^{2/(n-2)}$ gives,

$$\tilde{\mathcal{R}}_{[\nu\mu]}+2\mathcal{A}_{[\nu,\mu]}\sqrt{-2\Lambda_b}+2\Lambda_b[\mathcal{A}_\nu,\mathcal{A}_\mu]+\Lambda_bN_{[\nu\mu]}=0. \qquad (7.52)$$

Note that the antisymmetric field equations (7.52) lack a source term because $\mathcal{L}_m$ in (7.20) contains only $\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}=N^{1/2d}N^{\dashv(\nu\mu)}$ from (7.10), and not $N^{1/2d}N^{\dashv[\nu\mu]}$. The trace operations in (7.49,7.50) occur because we are assuming the special case (7.46). The off-diagonal matrix components of $\delta\mathcal{L}/\delta\tilde\Gamma^\beta_{\tau\rho}$ and $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv(\mu\nu)})$ vanish because with (7.46), the Lagrangian density contains no off-diagonal matrix components of $\tilde\Gamma^\beta_{\tau\rho}$ and $N^{1/2d}N^{\dashv(\mu\nu)}$. The trace operation sums up the contributions from the diagonal matrix components of $\tilde\Gamma^\beta_{\tau\rho}$ and $N^{1/2d}N^{\dashv(\mu\nu)}$ because (7.46) means that for a given set of tensor indices, all of the diagonal matrix components are really the same variable.

To put (7.47-7.52) into a form which looks more like the ordinary Einstein-Weinberg-Salam field equations we need to do some preliminary calculations. The definitions (7.10,7.13) of $\mathsf{g}_{\nu\mu}$ and $f_{\nu\mu}$ can be inverted to give $N_{\nu\mu}$ in terms of $\mathsf{g}_{\nu\mu}$ and $f_{\nu\mu}$. An expansion in powers of Λ_b^{-1} is derived in Appendix C,

$$N_{(\nu\mu)} = \mathsf{g}_{\nu\mu} - 2\left(f^{\sigma}{}_{(\nu}f_{\mu)\sigma} - \frac{1}{2(n-2)}\,\mathsf{g}_{\nu\mu}\frac{tr(f^{\rho}{}_{\sigma}f^{\sigma}{}_{\rho})}{d}\right)\Lambda_b^{-1} + (f^3)\Lambda_b^{-3/2}\dots \quad (7.53)$$

$$N_{[\nu\mu]} = f_{\nu\mu}\sqrt{2}\,i\Lambda_b^{-1/2} + (f^2)\Lambda_b^{-1}\dots. \quad (7.54)$$

Here $(f^3)\Lambda_b^{-3/2}$ and $(f^2)\Lambda_b^{-1}$ refer to terms like $\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{(\mu}\hat{f}_{\nu)\rho}\Lambda_b^{-3/2}$ and $f^{\sigma}{}_{[\nu}f_{\mu]\sigma}\Lambda_b^{-1}$.

Because of the assumption (7.46) and the trace operation in (7.49), the connection equations (7.49) are the same as with the Abelian theory (2.55) but with the substitution of $tr[f_{\nu\mu}]/d$ and $tr[j^{\nu}]/d$ instead of $f_{\nu\mu}$ and j^{ν}. Therefore the solution of the connection equations from (2.38) can again be abbreviated as

$$\tilde{\Gamma}^{\alpha}_{(\nu\mu)} = I\Gamma^{\alpha}_{\nu\mu} + (f'f)\Lambda_b^{-1}\dots \qquad \tilde{\Gamma}^{\alpha}_{[\nu\mu]} = (f')\Lambda_b^{-1}\dots, \quad (7.55)$$

where $\Gamma^{\alpha}_{\nu\mu}$ is the Christoffel connection,

$$\Gamma^{\alpha}_{\nu\mu} = \frac{1}{2}\,g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\nu\mu,\sigma}). \quad (7.56)$$

Substituting (7.55) using (R.4) shows that as in (2.39), the Non-symmetric Ricci tensor (7.21) can again be abbreviated as

$$\tilde{\mathcal{R}}_{(\nu\mu)} = IR_{\nu\mu} + (f'f')\Lambda_b^{-1} + (ff'')\Lambda_b^{-1}\dots, \qquad \tilde{\mathcal{R}}_{[\nu\mu]} = (f'')\Lambda_b^{-1/2}\dots, \quad (7.57)$$

where $R_{\nu\mu} = R_{\nu\mu}(\Gamma)$ is the ordinary Ricci tensor. Here $(f'f')\Lambda_b^{-1}$, $(ff'')\Lambda_b^{-1}$ and $(f'')\Lambda_b^{-1/2}$ indicate terms like $tr(f^{\sigma}{}_{\nu;\alpha})tr(f^{\alpha}{}_{\mu;\sigma})\Lambda_b^{-1}$, $tr(f^{\alpha\tau})tr(f_{\tau(\nu;\mu);\alpha})\Lambda_b^{-1}$ and $tr(f_{[\nu\mu,\alpha];}{}^{\alpha})\Lambda_b^{-1/2}$.

Combining (7.53,7.57,2.3) with the symmetric field equations (7.50) and their contraction gives

$$\begin{aligned}G_{\nu\mu} &= 8\pi\frac{tr(T_{\nu\mu})}{d}+2\left(\frac{tr(f^{\sigma}{}_{(\nu}f_{\mu)\sigma})}{d}-\frac{1}{4}\,g_{\nu\mu}\frac{tr(f^{\rho\sigma}f_{\sigma\rho})}{d}\right)\\ &\quad+\Lambda\left(\frac{n}{2}-1\right)g_{\nu\mu}+(f^3)\Lambda_b^{-1/2}+(f'f')\Lambda_b^{-1}+(ff'')\Lambda_b^{-1}\dots,\end{aligned}\qquad (7.58)$$

where the Einstein tensor and energy-momentum tensor are

$$G_{\nu\mu} = R_{\nu\mu}-\frac{1}{2}g_{\nu\mu}\,R^{\alpha}_{\alpha},\qquad T_{\nu\mu}=S_{\nu\mu}-\frac{1}{2}\,g_{\nu\mu}S^{\alpha}_{\alpha}. \qquad (7.59)$$

Here $(f^3)\Lambda_b^{-1/2}$, $(f'f')\Lambda_b^{-1}$ and $(ff'')\Lambda_b^{-1}$ indicate terms like $f^{\rho}{}_{\sigma}f^{\sigma}{}_{(\mu}f_{\nu)\rho}\Lambda_b^{-1/2}$, $tr(f^{\sigma}{}_{\nu;\alpha})tr(f^{\alpha}{}_{\mu;\sigma})\Lambda_b^{-1}$ and $tr(f^{\alpha\tau})tr(f_{\tau(\nu;\,\mu);\alpha})\Lambda_b^{-1}$. This shows that the Einstein equations (7.58) match those of Einstein-Weinberg-Salam theory except for extra terms which will be very small relative to the leading order terms because of the large value $\Lambda_b\sim 10^{63}cm^{-2}$ from (2.12).

Combining (7.54,7.57) with the antisymmetric field equations (7.52) gives

$$f_{\nu\mu}=2\mathcal{A}_{[\mu,\nu]}+\sqrt{-2\Lambda_b}\,[\mathcal{A}_\nu,\mathcal{A}_\mu]+(f^2)\Lambda_b^{-1/2}+(f'')\Lambda_b^{-1}\dots. \qquad (7.60)$$

Here $(f^2)\Lambda_b^{-1/2}$ and $(f'')\Lambda_b^{-1}$ indicate terms like $f^{\sigma}{}_{[\nu}f_{\mu]\sigma}\Lambda_b^{-1}$ and $tr(f_{[\nu\mu,\alpha];}{}^{\alpha})\Lambda_b^{-1/2}$. From (2.12) we see that the $f_{\nu\mu}$ in Ampere's law (7.47) matches the electro-weak tensor (7.2) except for extra terms which will be very small relative to the leading order terms because of the large value $\Lambda_b\!\sim\!10^{63}cm^{-2}$ from (2.12).

Finally, let us do a quantitative comparison of our non-Abelian LRES theory to Einstein-Weinberg-Salam theory. If Λ_z is due to zero-point fluctuations we would usually expect $\Lambda_b\!\sim\!\omega_c^4 l_P^2\!\sim\!10^{66}cm^{-2}$ with cutoff frequency $\omega_c\!\sim\!1/l_P$ as in (2.12,2.13). Our Λ_b from (7.15) is consistent with this interpretation with a cutoff frequency

$\omega_c \sim \alpha^{1/4}/l_P$, which is just as reasonable as $\omega_c \sim 1/l_P$ as far as anyone knows. For the Abelian LRES theory with $\Lambda_b \sim 10^{66} cm^{-2}$, we showed in §2.4 that the higher order terms in the Einstein-Maxwell field equations were $< 10^{-16}$ of the ordinary terms for worst-case field strengths and rates of change accessible to measurement. Therefore, for non-Abelian LRES theory with $\Lambda_b = 1.457 \times 10^{63} cm^{-2}$ from (7.15), the higher order terms in the field equations will be $< 10^{-13}$ of the ordinary terms for worst-case field strengths and rates of change accessible to measurement. This is far below the level that could be detected by experiment.

One aspect of this theory which might differ from Einstein-Weinberg-Salam theory is the possible existence of Proca waves, as discussed at the end of §2.4 for the purely electromagnetic case. The only change for the non-Abelian case is that Λ_b is fixed, so we cannot use the argument that the potential ghost goes away in the limit as $\omega_c \to \infty,\ \Lambda_b \to \infty$. If Proca-waves really do exist in the theory, it is possible that they could be interpreted as a built-in Pauli-Villars field as discussed in §2.4 and Appendix K. Finally, we should mention again that this theory would differ from Einstein-Weinberg-Salam theory if we do not assume the special case (7.46) where $\mathsf{g}^{\nu\mu}$ and $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ are restricted to be multiples of the identity matrix. Further work is necessary to compare this more general theory to experiment for reasonable choices of the metric definition (7.11). Some preliminary work on this topic can be found in Appendix X.

Chapter 8

Conclusions

The Einstein-Schrödinger theory is modified to include a cosmological constant Λ_z which multiplies the symmetric metric. This cosmological constant is assumed to be nearly cancelled by Schrödinger's "bare" cosmological constant Λ_b which multiplies the nonsymmetric fundamental tensor, such that the total "physical" cosmological constant $\Lambda = \Lambda_b + \Lambda_z$ matches measurement. The resulting Λ-renormalized Einstein-Schrödinger theory closely approximates ordinary Einstein-Maxwell theory when $|\Lambda_z| \sim 1/(\text{Planck length})^2$, and it becomes exactly Einstein-Maxwell theory in the limit as $|\Lambda_z| \to \infty$. In a similar manner, when the theory is generalized to non-Abelian fields, a special case closely approximates Einstein-Weinberg-Salam theory.

Appendix A

A divergence identity

Here we derive (4.4) using only the definitions (2.4,2.22) of $g_{\nu\mu}$ and $f_{\nu\mu}$, and the identity (2.56),

$$\left(N^{(\mu}{}_{\nu)}-\frac{1}{2}\delta^\mu_\nu N^\rho_\rho\right)_{;\,\mu}-\frac{3}{2}f^{\sigma\rho}N_{[\sigma\rho,\nu]}\sqrt{2}\,i\Lambda_b^{-1/2} \tag{A.1}$$

$$=\frac{1}{2}g^{\sigma\rho}(N_{(\rho\nu);\sigma}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})-\frac{3}{2}f^{\sigma\rho}N_{[\sigma\rho;\nu]}\sqrt{2}\,i\Lambda_b^{-1/2} \tag{A.2}$$

$$=\frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}\left[N^{\dashv(\sigma\rho)}(N_{(\rho\nu);\sigma}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})-3N^{\dashv[\rho\sigma]}N_{[\sigma\rho;\nu]}\right] \tag{A.3}$$

$$=\frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}\left[N^{\dashv\sigma\rho}(N_{(\rho\nu);\sigma}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})+3N^{\dashv\sigma\rho}N_{[\rho\nu;\sigma]}\right] \tag{A.4}$$

$$=\frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv\sigma\rho}(N_{\rho\nu;\sigma}+N_{\nu\sigma;\rho}-N_{\rho\sigma;\nu}) \tag{A.5}$$

$$=\frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}\left[N^{\dashv\sigma\rho}(N_{\rho\nu;\sigma}+N_{\nu\sigma;\rho})-N^{\dashv\sigma\rho}(N_{\rho\sigma,\nu}-\Gamma^\alpha_{\rho\nu}N_{\alpha\sigma}-\Gamma^\alpha_{\sigma\nu}N_{\rho\alpha})\right] \tag{A.6}$$

$$=-\frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}(N^{\dashv\sigma\rho}{}_{;\sigma}N_{\rho\nu}+N^{\dashv\sigma\rho}{}_{;\rho}N_{\nu\sigma})-\frac{1}{\sqrt{-g}}(\sqrt{-N}\,)_{;\nu} \tag{A.7}$$

$$=-\frac{1}{2}\left[\left(\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv\sigma\rho}\right)_{;\sigma}N_{\rho\nu}+\left(\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv\sigma\rho}\right)_{;\rho}N_{\nu\sigma}\right] \tag{A.8}$$

$$=-\frac{1}{2}\left[(g^{\rho\sigma}+f^{\rho\sigma}\sqrt{2}\,i\Lambda_b^{-1/2})_{;\sigma}N_{\rho\nu}+(g^{\rho\sigma}+f^{\rho\sigma}\sqrt{2}\,i\Lambda_b^{-1/2})_{;\rho}N_{\nu\sigma}\right] \tag{A.9}$$

$$=f^{\sigma\rho}{}_{;\sigma}N_{[\rho\nu]}\sqrt{2}\,i\Lambda_b^{-1/2}. \tag{A.10}$$

Appendix B

Variational derivatives for fields with the symmetry $\tilde{\Gamma}^{\sigma}_{[\mu\sigma]}=0$

The field equations associated with a field with symmetry properties must have the same number of independent components as the field. For a field with the symmetry $\tilde{\Gamma}^{\sigma}_{[\mu\sigma]} = 0$, the field equations can be found by introducing a Lagrange multiplier Ω^μ,

$$0 = \delta \int (\mathcal{L} + \Omega^\mu \tilde{\Gamma}^{\sigma}_{[\mu\sigma]}) d^n x. \tag{B.1}$$

Minimizing the integral with respect to Ω^μ shows that the symmetry is enforced. Using the definition,

$$\frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^{\beta}_{\tau\rho}} = \frac{\partial\mathcal{L}}{\partial\tilde{\Gamma}^{\beta}_{\tau\rho}} - \left(\frac{\partial\mathcal{L}}{\partial\tilde{\Gamma}^{\beta}_{\tau\rho,\omega}}\right)_{,\omega} \cdots\, , \tag{B.2}$$

and minimizing the integral with respect to $\tilde{\Gamma}^{\beta}_{\tau\rho}$ gives

$$0 = \frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^{\beta}_{\tau\rho}} + \Omega^\mu \delta^\sigma_\beta \delta^\tau_{[\mu} \delta^\rho_{\sigma]} = \frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^{\beta}_{\tau\rho}} + \frac{1}{2}(\Omega^\tau \delta^\rho_\beta - \delta^\tau_\beta \Omega^\rho). \tag{B.3}$$

Contracting this on the left and right gives

$$\Omega^\rho = \frac{2}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^\alpha_{\alpha\rho}} = -\frac{2}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^\alpha_{\rho\alpha}}. \tag{B.4}$$

Substituting (B.4) back into (B.3) gives

$$0 \;=\; \frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^\beta_{\tau\rho}} - \frac{\delta^\tau_\beta}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^\alpha_{\alpha\rho}} - \frac{\delta^\rho_\beta}{(n-1)}\frac{\Delta\mathcal{L}}{\Delta\tilde{\Gamma}^\alpha_{\tau\alpha}}. \tag{B.5}$$

In (B.4,B.5) the index contractions occur after the derivatives. Contracting (B.5) on the right and left gives the same result, so it has the same number of independent components as $\tilde{\Gamma}^\alpha_{\mu\nu}$. This is a general expression for the field equations associated with a field having the symmetry $\tilde{\Gamma}^\sigma_{[\mu\sigma]} = 0$.

Appendix C

Approximate solution for $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$

Here we invert the definitions (7.11,7.13) of $g_{\nu\mu}$ and $f_{\nu\mu}$ to obtain (7.53,7.54), the approximation of $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$, and we also do the same for Abelian fields as in (2.4,2.22) and (2.34,2.35). First let us define the notation

$$\hat{f}^{\nu\mu} = f^{\nu\mu}\sqrt{2}\, i\, \Lambda_b^{-1/2}. \tag{C.1}$$

We assume that $|\hat{f}^{\nu}{}_{\mu}| \ll 1$ for all components of the unitless field $\hat{f}^{\nu}{}_{\mu}$, and find a solution in the form of a power series expansion in $\hat{f}^{\nu}{}_{\mu}$.

We will first consider the problem for non-Abelian fields. For the following calculations we will treat the fields as $nd{\times}nd$ matrices but we will only show the tensor indices explicitly. Lowering an index on the right side of the equation $(N/\mathsf{g})^{1/2d} N^{\dashv\nu\mu} = \mathsf{g}^{\mu\nu} + \hat{f}^{\mu\nu}$ from (7.14) we get

$$(N/\mathsf{g})^{1/2d} N^{\dashv\mu}{}_{\alpha} = \delta^{\mu}_{\alpha} I - \hat{f}^{\mu}{}_{\alpha}. \tag{C.2}$$

Using $\hat{f}^{\alpha}{}_{\alpha}\!=\!0$, the well known formula $det(e^M) = exp\,(tr(M))$, and the power series $ln(1\!-\!x) = -x - x^2/2 - x^3/3\ldots$ we get[85],

$$ln(det(I\!-\!\hat{f})) \;\;=\;\; tr(ln(I\!-\!\hat{f})) = -\frac{1}{2}\,tr(\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}) + (\hat{f}^3)\ldots \qquad \text{(C.3)}$$

Here the notation $(\hat{f}^3)$ refers to terms like $tr(\hat{f}^{\tau}{}_{\alpha}\hat{f}^{\alpha}{}_{\sigma}\hat{f}^{\sigma}{}_{\tau})$. Taking $ln(det())$ on both sides of (C.2) using the result (C.3) and the identities $det(sM) = s^{nd}det(M)$ and $det(M^{-1})\!=1/det(M)$ gives

$$ln(det[(N/\mathsf{g})^{1/2d}N^{\dashv\mu}{}_{\alpha}]) = ln((N/\mathsf{g})^{n/2-1}) = -\frac{1}{2}\,tr(\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}) + (\hat{f}^3)\ldots, \qquad \text{(C.4)}$$

$$ln[(N/\mathsf{g})^{1/2d}] = -\frac{1}{2d(n\!-\!2)}\,tr(\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}) + (\hat{f}^3)\ldots. \qquad \text{(C.5)}$$

Taking e^x on both sides of this and using $e^x = 1 + x + x^2/2\ldots$ gives

$$(N/\mathsf{g})^{1/2d} = 1\!-\!\frac{1}{2d(n\!-\!2)}\,tr(\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}) + (\hat{f}^3)\ldots. \qquad \text{(C.6)}$$

Using the power series $(1\!-\!x)^{-1}\!=\!1 + x + x^2 + x^3\ldots$, or multiplying by (C.2) on the right we can calculate the inverse of (C.2) to get[85]

$$(\mathsf{g}/N)^{1/2d}N^{\nu}{}_{\mu} = \delta^{\nu}_{\mu}I + \hat{f}^{\nu}{}_{\mu} + \hat{f}^{\nu}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu} + (\hat{f}^3)\ldots. \qquad \text{(C.7)}$$

Lowering this on the left gives,

$$N_{\nu\mu} = (N/\mathsf{g})^{1/2d}(\mathsf{g}_{\nu\mu} + \hat{f}_{\nu\mu} + \hat{f}_{\nu\sigma}\hat{f}^{\sigma}{}_{\mu} + (\hat{f}^3)\ldots). \qquad \text{(C.8)}$$

Here $(\hat{f}^3)$ refers to terms like $\hat{f}_{\nu\alpha}\hat{f}^{\alpha}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu}$. Using (7.46,C.8,C.6,C.1) we get the result (7.53,7.54).

Now let us redo the calculation for Abelian fields. Lowering an index on the equation $(\sqrt{-N}/\sqrt{-g}\,)N^{\dashv\mu\nu}\!= g^{\nu\mu}\!+ \hat{f}^{\nu\mu}$ from (2.4,2.22) gives

$$\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv\mu}{}_{\alpha} = \delta^{\mu}_{\alpha} - \hat{f}^{\mu}{}_{\alpha}. \qquad \text{(C.9)}$$

Let us consider the tensor $\hat{f}^{\mu}{}_{\alpha} = \hat{f}^{\mu\nu} g_{\nu\alpha}$. Because $g_{\nu\alpha}$ is symmetric and $\hat{f}^{\mu\nu}$ is antisymmetric, it is clear that $\hat{f}^{\alpha}{}_{\alpha} = 0$. Also because $\hat{f}_{\nu\sigma}\hat{f}^{\sigma}{}_{\mu}$ is symmetric it is clear that $\hat{f}^{\nu}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu}\hat{f}^{\mu}{}_{\nu} = 0$. In matrix language therefore $tr(\hat{f}) = 0$, $tr(\hat{f}^3) = 0$, and in fact $tr(\hat{f}^p) = 0$ for any odd p. Using the well known formula $det(e^M) = exp\,(tr(M))$ and the power series $ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4\ldots$ we then get[85],

$$ln(det(I-\hat{f})) \;=\; tr(ln(I-\hat{f})) = -\frac{1}{2}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho} + (\hat{f}^4)\ldots \tag{C.10}$$

Here the notation $(\hat{f}^4)$ refers to terms like $\hat{f}^{\tau}{}_{\alpha}\hat{f}^{\alpha}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}\hat{f}^{\rho}{}_{\tau}$. Taking $ln(det())$ on both sides of (C.9) using the result (C.10) and the identities $det(sM) = s^n det(M)$ and $det(M^{-1}) = 1/det(M)$ gives

$$ln\left(\frac{\sqrt{-N}}{\sqrt{-g}}\right) \;=\; \frac{1}{(n-2)}\,ln\left(\frac{N^{(n/2-1)}}{g^{(n/2-1)}}\right) = -\frac{1}{2(n-2)}\,\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho} + (\hat{f}^4)\ldots \tag{C.11}$$

Taking e^x on both sides of (C.11) and using $e^x = 1 + x + x^2/2\ldots$ gives

$$\frac{\sqrt{-N}}{\sqrt{-g}} = 1 - \frac{1}{2(n-2)}\,\hat{f}^{\rho\sigma}\hat{f}_{\sigma\rho} + (\hat{f}^4)\ldots \tag{C.12}$$

Using the power series $(1-x)^{-1} = 1 + x + x^2 + x^3\ldots$, or multiplying (C.9) term by term, we can calculate the inverse of (C.9) to get[85]

$$\frac{\sqrt{-g}}{\sqrt{-N}}N^{\nu}{}_{\mu} = \delta^{\nu}_{\mu} + \hat{f}^{\nu}{}_{\mu} + \hat{f}^{\nu}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu} + \hat{f}^{\nu}{}_{\rho}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu} + (\hat{f}^4)\ldots \tag{C.13}$$

$$N_{\nu\mu} = \frac{\sqrt{-N}}{\sqrt{-g}}(g_{\nu\mu} + \hat{f}_{\nu\mu} + \hat{f}_{\nu\sigma}\hat{f}^{\sigma}{}_{\mu} + \hat{f}_{\nu\rho}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu} + (\hat{f}^4)\ldots). \tag{C.14}$$

Here the notation $(\hat{f}^4)$ refers to terms like $\hat{f}_{\nu\alpha}\hat{f}^{\alpha}{}_{\sigma}\hat{f}^{\sigma}{}_{\rho}\hat{f}^{\rho}{}_{\mu}$. Since $\hat{f}_{\nu\sigma}\hat{f}^{\sigma}{}_{\mu}$ is symmetric and $\hat{f}_{\nu\rho}\hat{f}^{\rho}{}_{\sigma}\hat{f}^{\sigma}{}_{\mu}$ is antisymmetric, we obtain from (C.14,C.12,C.1) the final result (2.34,2.35).

Appendix D

Approximate solution for $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$

Here we derive the approximate solution (2.62,2.63) to the connection equations (2.55). First let us define the notation

$$\hat{f}^{\nu\mu}\!=\!f^{\nu\mu}\sqrt{2}\,i\,\Lambda_b^{-1/2},\quad \hat{j}^{\sigma}\!=\!j^{\sigma}\sqrt{2}\,i\,\Lambda_b^{-1/2},\quad \bar{\Upsilon}^{\alpha}_{\nu\mu}\!=\!\Upsilon^{\alpha}_{(\nu\mu)},\quad \check{\Upsilon}^{\alpha}_{\nu\mu}\!=\!\Upsilon^{\alpha}_{[\nu\mu]}. \tag{D.1}$$

We assume that $|\hat{f}^{\nu}{}_{\mu}|\ll 1$ for all components of the unitless field $\hat{f}^{\nu}{}_{\mu}$, and find a solution in the form of a power series expansion in $\hat{f}^{\nu}{}_{\mu}$. Using (2.59) and

$$\tilde{\Gamma}^{\sigma}_{\sigma\alpha} \;=\; \frac{(\sqrt{-N})_{,\alpha}}{\sqrt{-N}}+\frac{8\pi}{(n\!-\!2)(n\!-\!1)}\frac{\sqrt{-g}}{\sqrt{-N}}\hat{j}^{\sigma}N_{[\sigma\alpha]} \tag{D.2}$$

from (2.57) and $\tilde\Gamma^\alpha_{\nu\mu}=\Gamma^\alpha_{\nu\mu}+\Upsilon^\alpha_{\nu\mu}$, $(\sqrt{-N}/\sqrt{-g}\,)N^{\dashv\mu\nu}=g^{\nu\mu}+\hat f^{\nu\mu}$ from (2.61,2.4,2.22) we get

$$
\begin{aligned}
0 &= \frac{\sqrt{-N}}{\sqrt{-g}}(N^{\dashv\mu\nu}{}_{,\alpha}+\tilde\Gamma^\nu_{\tau\alpha}N^{\dashv\mu\tau}+\tilde\Gamma^\mu_{\alpha\tau}N^{\dashv\tau\nu})\\
&\quad -\frac{8\pi}{(n\!-\!1)}\left(\hat\jmath^{[\mu}\delta^{\nu]}_\alpha+\frac{1}{(n\!-\!2)}\hat\jmath^\tau N_{[\tau\alpha]}N^{\dashv\mu\nu}\right) \qquad (D.3)\\
&= \left(\frac{\sqrt{-N}N^{\dashv\mu\nu}}{\sqrt{-g}}\right)_{,\alpha}+\frac{\sqrt{-N}}{\sqrt{-g}}(\tilde\Gamma^\nu_{\tau\alpha}N^{\dashv\mu\tau}+\tilde\Gamma^\mu_{\alpha\tau}N^{\dashv\tau\nu}-(\tilde\Gamma^\sigma_{\sigma\alpha}-\Gamma^\sigma_{\sigma\alpha})N^{\dashv\mu\nu})\\
&\quad -\frac{8\pi}{(n\!-\!1)}\hat\jmath^{[\mu}\delta^{\nu]}_\alpha \qquad (D.4)\\
&= (g^{\nu\mu}+\hat f^{\nu\mu})_{;\alpha}+\Upsilon^\nu_{\tau\alpha}(g^{\tau\mu}+\hat f^{\tau\mu})+\Upsilon^\mu_{\alpha\tau}(g^{\nu\tau}+\hat f^{\nu\tau})-\Upsilon^\sigma_{\sigma\alpha}(g^{\nu\mu}+\hat f^{\nu\mu})\\
&\quad -\frac{8\pi}{(n\!-\!1)}\hat\jmath^{[\mu}\delta^{\nu]}_\alpha \qquad (D.5)\\
&= \hat f^{\nu\mu}{}_{;\alpha}+\Upsilon^\nu_{\tau\alpha}g^{\tau\mu}+\Upsilon^\nu_{\tau\alpha}\hat f^{\tau\mu}+\Upsilon^\mu_{\alpha\tau}g^{\nu\tau}+\Upsilon^\mu_{\alpha\tau}\hat f^{\nu\tau}-\Upsilon^\sigma_{\sigma\alpha}g^{\nu\mu}-\Upsilon^\sigma_{\sigma\alpha}\hat f^{\nu\mu}\\
&\quad +\frac{4\pi}{(n\!-\!1)}(\hat\jmath^\nu\delta^\mu_\alpha-\hat\jmath^\mu\delta^\nu_\alpha). \qquad (D.6)
\end{aligned}
$$

Contracting this with $g_{\nu\mu}$ gives

$$
0=(2-n)\Upsilon^\sigma_{\sigma\alpha}-2\check\Upsilon^\sigma_{\alpha\tau}\hat f^\tau{}_\sigma \quad\Rightarrow\quad \Upsilon^\sigma_{\sigma\alpha}=\frac{2}{(n\!-\!2)}\check\Upsilon_{\sigma\tau\alpha}\hat f^{\tau\sigma}. \qquad (D.7)
$$

Lowering the indices of (D.6) and making linear combinations of its permutations gives

$$
\begin{aligned}
\Upsilon_{\alpha\nu\mu}=\Upsilon_{\alpha\nu\mu} &+ \frac12\bigg(\hat f_{\nu\mu;\alpha}+\Upsilon_{\nu\mu\alpha}+\Upsilon_{\nu\tau\alpha}\hat f^\tau{}_\mu+\Upsilon_{\mu\alpha\nu}+\Upsilon_{\mu\alpha\tau}\hat f_\nu{}^\tau-\Upsilon^\sigma_{\sigma\alpha}g_{\nu\mu}-\Upsilon^\sigma_{\sigma\alpha}\hat f_{\nu\mu}\\
&\qquad +\frac{4\pi}{(n\!-\!1)}(\hat\jmath_\nu g_{\alpha\mu}-\hat\jmath_\mu g_{\nu\alpha})\bigg)\\
&- \frac12\bigg(\hat f_{\mu\alpha;\nu}+\Upsilon_{\mu\alpha\nu}+\Upsilon_{\mu\tau\nu}\hat f^\tau{}_\alpha+\Upsilon_{\alpha\nu\mu}+\Upsilon_{\alpha\nu\tau}\hat f_\mu{}^\tau-\Upsilon^\sigma_{\sigma\nu}g_{\mu\alpha}-\Upsilon^\sigma_{\sigma\nu}\hat f_{\mu\alpha}\\
&\qquad +\frac{4\pi}{(n\!-\!1)}(\hat\jmath_\mu g_{\nu\alpha}-\hat\jmath_\alpha g_{\mu\nu})\bigg)\\
&- \frac12\bigg(\hat f_{\alpha\nu;\mu}+\Upsilon_{\alpha\nu\mu}+\Upsilon_{\alpha\tau\mu}\hat f^\tau{}_\nu+\Upsilon_{\nu\mu\alpha}+\Upsilon_{\nu\mu\tau}\hat f_\alpha{}^\tau-\Upsilon^\sigma_{\sigma\mu}g_{\alpha\nu}-\Upsilon^\sigma_{\sigma\mu}\hat f_{\alpha\nu}\\
&\qquad +\frac{4\pi}{(n\!-\!1)}(\hat\jmath_\alpha g_{\mu\nu}-\hat\jmath_\nu g_{\alpha\mu})\bigg). \qquad (D.8)
\end{aligned}
$$

Cancelling out the $\Upsilon_{\alpha\nu\mu}$ terms on the right-hand side, collecting terms, and separating out the symmetric and antisymmetric parts gives,

$$\bar{\Upsilon}_{\alpha\nu\mu} = \check{\Upsilon}_{[\alpha\mu]\tau}\hat{f}^{\tau}{}_{\nu} + \check{\Upsilon}_{[\alpha\nu]\tau}\hat{f}^{\tau}{}_{\mu} + \check{\Upsilon}_{(\nu\mu)\tau}\hat{f}^{\tau}{}_{\alpha} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\alpha}g_{\nu\mu} + \Upsilon^{\sigma}_{\sigma(\nu}g_{\mu)\alpha} \tag{D.9}$$

$$\check{\Upsilon}_{\alpha\nu\mu} = -\bar{\Upsilon}_{(\alpha\mu)\tau}\hat{f}^{\tau}{}_{\nu} + \bar{\Upsilon}_{(\alpha\nu)\tau}\hat{f}^{\tau}{}_{\mu} + \bar{\Upsilon}_{[\nu\mu]\tau}\hat{f}^{\tau}{}_{\alpha} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\alpha}\hat{f}_{\nu\mu} + \Upsilon^{\sigma}_{\sigma[\nu}\hat{f}_{\mu]\alpha}$$
$$+\frac{1}{2}(\hat{f}_{\nu\mu;\alpha} + \hat{f}_{\alpha\mu;\nu} - \hat{f}_{\alpha\nu;\mu}) + \frac{8\pi}{(n-1)}\hat{j}_{[\nu}g_{\mu]\alpha}. \tag{D.10}$$

Substituting (D.9) into (D.10)

$$\begin{aligned}
\check{\Upsilon}_{\alpha\nu\mu} &= -\frac{1}{2}\left(\check{\Upsilon}_{[\alpha\tau]\sigma}\hat{f}^{\sigma}{}_{\mu} + \check{\Upsilon}_{[\alpha\mu]\sigma}\hat{f}^{\sigma}{}_{\tau} + \check{\Upsilon}_{(\mu\tau)\sigma}\hat{f}^{\sigma}{}_{\alpha} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\alpha}g_{\mu\tau} + \Upsilon^{\sigma}_{\sigma(\mu}g_{\tau)\alpha}\right)\hat{f}^{\tau}{}_{\nu} \\
&\quad -\frac{1}{2}\left(\check{\Upsilon}_{[\mu\tau]\sigma}\hat{f}^{\sigma}{}_{\alpha} + \check{\Upsilon}_{[\mu\alpha]\sigma}\hat{f}^{\sigma}{}_{\tau} + \check{\Upsilon}_{(\alpha\tau)\sigma}\hat{f}^{\sigma}{}_{\mu} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\mu}g_{\alpha\tau} + \Upsilon^{\sigma}_{\sigma(\alpha}g_{\tau)\mu}\right)\hat{f}^{\tau}{}_{\nu} \\
&\quad +\frac{1}{2}\left(\check{\Upsilon}_{[\alpha\tau]\sigma}\hat{f}^{\sigma}{}_{\nu} + \check{\Upsilon}_{[\alpha\nu]\sigma}\hat{f}^{\sigma}{}_{\tau} + \check{\Upsilon}_{(\nu\tau)\sigma}\hat{f}^{\sigma}{}_{\alpha} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\alpha}g_{\nu\tau} + \Upsilon^{\sigma}_{\sigma(\nu}g_{\tau)\alpha}\right)\hat{f}^{\tau}{}_{\mu} \\
&\quad +\frac{1}{2}\left(\check{\Upsilon}_{[\nu\tau]\sigma}\hat{f}^{\sigma}{}_{\alpha} + \check{\Upsilon}_{[\nu\alpha]\sigma}\hat{f}^{\sigma}{}_{\tau} + \check{\Upsilon}_{(\alpha\tau)\sigma}\hat{f}^{\sigma}{}_{\nu} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\nu}g_{\alpha\tau} + \Upsilon^{\sigma}_{\sigma(\alpha}g_{\tau)\nu}\right)\hat{f}^{\tau}{}_{\mu} \\
&\quad +\frac{1}{2}\left(\check{\Upsilon}_{[\nu\tau]\sigma}\hat{f}^{\sigma}{}_{\mu} + \check{\Upsilon}_{[\nu\mu]\sigma}\hat{f}^{\sigma}{}_{\tau} + \check{\Upsilon}_{(\mu\tau)\sigma}\hat{f}^{\sigma}{}_{\nu} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\nu}g_{\mu\tau} + \Upsilon^{\sigma}_{\sigma(\mu}g_{\tau)\nu}\right)\hat{f}^{\tau}{}_{\alpha} \\
&\quad -\frac{1}{2}\left(\check{\Upsilon}_{[\mu\tau]\sigma}\hat{f}^{\sigma}{}_{\nu} + \check{\Upsilon}_{[\mu\nu]\sigma}\hat{f}^{\sigma}{}_{\tau} + \check{\Upsilon}_{(\nu\tau)\sigma}\hat{f}^{\sigma}{}_{\mu} - \frac{1}{2}\Upsilon^{\sigma}_{\sigma\mu}g_{\nu\tau} + \Upsilon^{\sigma}_{\sigma(\nu}g_{\tau)\mu}\right)\hat{f}^{\tau}{}_{\alpha} \\
&\quad -\frac{1}{2}\Upsilon^{\sigma}_{\sigma\alpha}\hat{f}_{\nu\mu} + \Upsilon^{\sigma}_{\sigma[\nu}\hat{f}_{\mu]\alpha} \\
&\quad +\frac{1}{2}(\hat{f}_{\nu\mu;\alpha} + \hat{f}_{\alpha\mu;\nu} - \hat{f}_{\alpha\nu;\mu}) + \frac{8\pi}{(n-1)}\hat{j}_{[\nu}g_{\mu]\alpha} \\
&= -\frac{1}{2}\left(\check{\Upsilon}_{\alpha\tau\sigma}\hat{f}^{\sigma}{}_{\mu} + \check{\Upsilon}_{\mu\tau\sigma}\hat{f}^{\sigma}{}_{\alpha}\right)\hat{f}^{\tau}{}_{\nu} \\
&\quad +\frac{1}{2}\left(\check{\Upsilon}_{\alpha\tau\sigma}\hat{f}^{\sigma}{}_{\nu} + \check{\Upsilon}_{\nu\tau\sigma}\hat{f}^{\sigma}{}_{\alpha}\right)\hat{f}^{\tau}{}_{\mu} \\
&\quad +\frac{1}{2}\left(\check{\Upsilon}_{\tau\mu\sigma}\hat{f}^{\sigma}{}_{\nu} + 2\check{\Upsilon}_{[\nu\mu]\sigma}\hat{f}^{\sigma}{}_{\tau} - \check{\Upsilon}_{\tau\nu\sigma}\hat{f}^{\sigma}{}_{\mu}\right)\hat{f}^{\tau}{}_{\alpha} \\
&\quad +\frac{1}{2}\Upsilon^{\sigma}_{\sigma\alpha}\hat{f}_{\mu\nu} + \Upsilon^{\sigma}_{\sigma\tau}\hat{f}^{\tau}{}_{[\mu}g_{\nu]\alpha} \\
&\quad +\frac{1}{2}(\hat{f}_{\nu\mu;\alpha} + \hat{f}_{\alpha\mu;\nu} - \hat{f}_{\alpha\nu;\mu}) + \frac{8\pi}{(n-1)}\hat{j}_{[\nu}g_{\mu]\alpha},
\end{aligned}$$

and using (D.7) gives,

$$\begin{aligned}\check{\Upsilon}_{\alpha\nu\mu} &= \check{\Upsilon}_{\alpha\sigma\tau}\hat{f}^{\sigma}{}_{\mu}\hat{f}^{\tau}{}_{\nu}+\check{\Upsilon}_{(\mu\sigma)\tau}\hat{f}^{\sigma}{}_{\alpha}\hat{f}^{\tau}{}_{\nu}-\check{\Upsilon}_{(\nu\sigma)\tau}\hat{f}^{\sigma}{}_{\alpha}\hat{f}^{\tau}{}_{\mu}+\check{\Upsilon}_{[\nu\mu]\sigma}\hat{f}^{\sigma}{}_{\tau}\hat{f}^{\tau}{}_{\alpha}\\ &+\frac{1}{(n-2)}\check{\Upsilon}_{\sigma\tau\alpha}\hat{f}^{\tau\sigma}\hat{f}_{\mu\nu}+\frac{2}{(n-2)}\check{\Upsilon}_{\sigma\rho\tau}\hat{f}^{\rho\sigma}\hat{f}^{\tau}{}_{[\mu}g_{\nu]\alpha}\\ &+\frac{1}{2}(\hat{f}_{\nu\mu;\alpha}+\hat{f}_{\alpha\mu;\nu}-\hat{f}_{\alpha\nu;\mu})+\frac{8\pi}{(n-1)}\hat{j}_{[\nu}g_{\mu]\alpha}. \end{aligned}\tag{D.11}$$

Equation (D.11) is useful for finding exact solutions to the connection equations because it consists of only $n^2(n-1)/2$ equations in the $n^2(n-1)/2$ unknowns $\check{\Upsilon}_{\alpha\nu\mu}$. Also, from (D.11) we can immediately see that

$$\check{\Upsilon}_{\alpha\nu\mu}=\frac{1}{2}(\hat{f}_{\nu\mu;\alpha}+\hat{f}_{\alpha\mu;\nu}-\hat{f}_{\alpha\nu;\mu})+\frac{4\pi}{(n-1)}(\hat{j}_{\nu}g_{\mu\alpha}-\hat{j}_{\mu}g_{\nu\alpha})+(\hat{f}^{3\prime})\ldots. \tag{D.12}$$

Here the notation $(\hat{f}^{3\prime})$ refers to terms like $\hat{f}_{\alpha\tau}\hat{f}^{\tau}{}_{\sigma}\hat{f}^{\sigma}{}_{[\nu;\mu]}$. With (D.12) as a starting point, one can calculate more accurate $\check{\Upsilon}_{\alpha\nu\mu}$ by recursively substituting the current $\check{\Upsilon}_{\alpha\nu\mu}$ into (D.11). Then this $\check{\Upsilon}_{\alpha\nu\mu}$ can be substituted into (D.7,D.9) to get $\bar{\Upsilon}_{\alpha\nu\mu}$. For our purposes (D.12) will be accurate enough. Substituting (D.12) into (D.7) we get

$$\check{\Upsilon}_{(\alpha\nu)\mu} = -\hat{f}_{\mu(\nu;\alpha)}+\frac{4\pi}{(n-1)}(\hat{j}_{(\nu}g_{\alpha)\mu}-\hat{j}_{\mu}g_{\nu\alpha})+(\hat{f}^{3\prime})\ldots, \tag{D.13}$$

$$\check{\Upsilon}_{[\alpha\nu]\mu} = \frac{1}{2}\hat{f}_{\nu\alpha;\mu}+\frac{4\pi}{(n-1)}\hat{j}_{[\nu}g_{\alpha]\mu}+(\hat{f}^{3\prime})\ldots, \tag{D.14}$$

$$\begin{aligned}\Upsilon^{\sigma}_{\sigma\alpha} &= \frac{2}{(n-2)}\left(\frac{1}{2}\hat{f}_{\tau\sigma;\alpha}+\frac{4\pi}{(n-1)}\hat{j}_{[\tau}g_{\sigma]\alpha}\right)\hat{f}^{\tau\sigma}+(\hat{f}^{4\prime})\ldots\\ &= \frac{-1}{2(n-2)}(\hat{f}^{\rho\sigma}\hat{f}_{\sigma\rho})_{,\alpha}+\frac{8\pi}{(n-1)(n-2)}\hat{j}^{\tau}\hat{f}_{\tau\alpha}+(\hat{f}^{4\prime})\ldots.\end{aligned}\tag{D.15}$$

Substituting these equations into (D.9) gives

$$
\begin{aligned}
\bar{\Upsilon}_{\alpha\nu\mu} &= -\left(\frac{1}{2}\hat{f}_{\alpha\mu;\tau} + \frac{2\pi}{(n\!-\!1)}(\hat{j}_\alpha g_{\mu\tau} - \hat{j}_\mu g_{\alpha\tau})\right)\hat{f}^\tau{}_\nu \\
&\quad + \left(\frac{1}{2}\hat{f}_{\nu\alpha;\tau} + \frac{2\pi}{(n\!-\!1)}(\hat{j}_\nu g_{\alpha\tau} - \hat{j}_\alpha g_{\nu\tau})\right)\hat{f}^\tau{}_\mu \\
&\quad + \left(-\hat{f}_{\tau(\mu;\nu)} + \frac{2\pi}{(n\!-\!1)}(\hat{j}_\mu g_{\nu\tau} + \hat{j}_\nu g_{\mu\tau} - 2\hat{j}_\tau g_{\mu\nu})\right)\hat{f}^\tau{}_\alpha \\
&\quad - \frac{1}{2}\left(\frac{-1}{2(n\!-\!2)}(\hat{f}^{\rho\sigma}\hat{f}_{\sigma\rho})_{,\alpha} + \frac{8\pi}{(n\!-\!1)(n\!-\!2)}\hat{j}^\tau\hat{f}_{\tau\alpha}\right)g_{\nu\mu} \\
&\quad + \left(\frac{-1}{2(n\!-\!2)}(\hat{f}^{\rho\sigma}\hat{f}_{\sigma\rho})_{,(\nu} + \frac{8\pi}{(n\!-\!1)(n\!-\!2)}\hat{j}^\tau\hat{f}_{\tau(\nu}\right)g_{\mu)\alpha} + (\hat{f}^{4\prime})\dots \\
&= \hat{f}^\tau{}_{(\nu}\hat{f}_{\mu)\alpha;\tau} + \hat{f}_\alpha{}^\tau\hat{f}_{\tau(\nu;\mu)} + \frac{1}{4(n\!-\!2)}\left((\hat{f}^{\rho\sigma}\hat{f}_{\sigma\rho})_{,\alpha}g_{\nu\mu} - 2(\hat{f}^{\rho\sigma}\hat{f}_{\sigma\rho})_{,(\nu}g_{\mu)\alpha}\right) \\
&\quad + \frac{4\pi}{(n\!-\!2)}\hat{j}^\tau\left(\hat{f}_{\alpha\tau}g_{\nu\mu} + \frac{2}{(n\!-\!1)}\hat{f}_{\tau(\nu}g_{\mu)\alpha}\right) + (\hat{f}^{4\prime})\dots. \qquad \text{(D.16)}
\end{aligned}
$$

Here the notation $(\hat{f}^{4\prime})$ refers to terms like $\hat{f}_{\alpha\tau}\hat{f}^\tau{}_\sigma\hat{f}^\sigma{}_\rho\hat{f}^\rho{}_{(\nu;\mu)}$. Raising the indices on (D.16,D.12,D.15) and using (D.1) gives the final result (2.62,2.63,2.64).

$$
\begin{aligned}
\bar{\Upsilon}^\alpha_{\nu\mu} &= \hat{f}^\tau{}_{(\nu}\hat{f}_{\mu)}{}^\alpha{}_{;\tau} + \hat{f}^{\alpha\tau}\hat{f}_{\tau(\nu;\mu)} + \frac{1}{4(n\!-\!2)}(\,\ell_,{}^\alpha g_{\nu\mu} - 2\,\ell_{,(\nu}\delta^\alpha_{\mu)}) \\
&\quad + \frac{4\pi}{(n\!-\!2)}\hat{j}^\rho\left(\hat{f}^\alpha{}_\rho g_{\nu\mu} + \frac{2}{(n\!-\!1)}\hat{f}_{\rho(\nu}\delta^\alpha_{\mu)}\right) + (\hat{f}^{4\prime})\dots, \qquad \text{(D.17)} \\
\check{\Upsilon}^\alpha_{\nu\mu} &= \frac{1}{2}(\hat{f}_{\nu\mu;}{}^\alpha + \hat{f}^\alpha{}_{\mu;\nu} - \hat{f}^\alpha{}_{\nu;\mu}) + \frac{8\pi}{(n\!-\!1)}\hat{j}_{[\nu}\delta^\alpha_{\mu]} + (\hat{f}^{3\prime})\dots. \qquad \text{(D.18)}
\end{aligned}
$$

Appendix E

Derivation of the generalized contracted Bianchi identity

Here we derive the generalized contracted Bianchi identity (4.3) from the connection equations (2.55), and from the symmetry (2.8) of $\tilde{\Gamma}^{\alpha}_{\nu\mu}$. Whereas [45] derived the identity by performing an infinitesimal coordinate transformation on an invariant integral, we will instead use a direct method similar to [3], but generalized to include charge currents. First we make the following definitions,

$$\mathbf{W}^{\tau\rho} = \sqrt{-g}\,W^{\tau\rho} = \sqrt{-N}N^{\dashv\rho\tau} = \sqrt{-g}\,(g^{\tau\rho} + \hat{f}^{\tau\rho}), \tag{E.1}$$

$$\hat{f}^{\nu\mu} = f^{\nu\mu}\sqrt{2}\,i\Lambda_b^{-1/2}, \quad \hat{\mathbf{j}}^{\alpha} = \sqrt{-g}j^{\alpha}\sqrt{2}\,i\Lambda_b^{-1/2}, \tag{E.2}$$

$$\tilde{\mathcal{R}}^{\tau}{}_{\nu\alpha\mu} = \tilde{\Gamma}^{\tau}_{\nu\mu,\alpha} - \tilde{\Gamma}^{\tau}_{\nu\alpha,\mu} + \tilde{\Gamma}^{\sigma}_{\nu\mu}\tilde{\Gamma}^{\tau}_{\sigma\alpha} - \tilde{\Gamma}^{\sigma}_{\nu\alpha}\tilde{\Gamma}^{\tau}_{\sigma\mu} + \delta^{\tau}_{\nu}\tilde{\Gamma}^{\sigma}_{\sigma[\alpha,\mu]}, \tag{E.3}$$

$$\tilde{\mathcal{R}}_{\nu\mu} = \tilde{\mathcal{R}}^{\alpha}{}_{\nu\alpha\mu} = \tilde{\Gamma}^{\alpha}_{\nu\mu,\alpha} - \tilde{\Gamma}^{\alpha}_{\nu\alpha,\mu} + \tilde{\Gamma}^{\sigma}_{\nu\mu}\tilde{\Gamma}^{\alpha}_{\sigma\alpha} - \tilde{\Gamma}^{\sigma}_{\nu\alpha}\tilde{\Gamma}^{\alpha}_{\sigma\mu} + \tilde{\Gamma}^{\sigma}_{\sigma[\nu,\mu]}. \tag{E.4}$$

Here $\tilde{\mathcal{R}}_{\nu\mu}$ is our non-symmetric Ricci tensor (2.11), which has the property from (2.16),

$$\mathcal{R}_{\nu\mu}(\tilde{\Gamma}^T) = \tilde{\mathcal{R}}_{\mu\nu}. \tag{E.5}$$

The tensors $\tilde{\mathcal{R}}_{\nu\mu}$ and $\tilde{\mathcal{R}}^{\tau}{}_{\nu\alpha\mu}$ reduce to the ordinary Ricci and Riemann tensors for symmetric fields where $\Gamma^{\sigma}_{\sigma[\nu,\mu]} = R^{\sigma}{}_{\sigma\mu\nu}/2 = 0$.

Rewriting the connection equations (2.55) in terms of the definitions above gives,

$$0 \;=\; \mathbf{W}^{\tau\rho}{}_{,\lambda} + \tilde{\Gamma}^{\tau}_{\sigma\lambda}\mathbf{W}^{\sigma\rho} + \tilde{\Gamma}^{\rho}_{\lambda\sigma}\mathbf{W}^{\tau\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma\lambda}\mathbf{W}^{\tau\rho} - \frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\rho}\delta^{\tau}_{\lambda} - \hat{\mathbf{j}}^{\tau}\delta^{\rho}_{\lambda}). \tag{E.6}$$

Differentiating (E.6), antisymmetrizing, and substituting (E.6) for $\mathbf{W}^{\tau\rho}{}_{,\lambda}$ gives,

$$\begin{aligned}
0 \;=&\; \left(\mathbf{W}^{\tau\rho}{}_{,[\lambda} + \tilde{\Gamma}^{\tau}_{\sigma[\lambda}\mathbf{W}^{\sigma\rho} + \tilde{\Gamma}^{\rho}_{[\lambda|\sigma}\mathbf{W}^{\tau\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma[\lambda}\mathbf{W}^{\tau\rho} - \frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\rho}\delta^{\tau}_{[\lambda} - \hat{\mathbf{j}}^{\tau}\delta^{\rho}_{[\lambda})\right)_{,\nu]} \qquad \text{(E.7)}\\
=&\; \tilde{\Gamma}^{\tau}_{\sigma[\lambda,\nu]}\mathbf{W}^{\sigma\rho} + \tilde{\Gamma}^{\rho}_{[\lambda|\sigma,|\nu]}\mathbf{W}^{\tau\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma[\lambda,\nu]}\mathbf{W}^{\tau\rho} - \frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\rho}{}_{,[\nu}\delta^{\tau}_{\lambda]} - \hat{\mathbf{j}}^{\tau}{}_{,[\nu}\delta^{\rho}_{\lambda]})\\
&+\tilde{\Gamma}^{\tau}_{\sigma[\lambda}\mathbf{W}^{\sigma\rho}{}_{,\nu]} + \tilde{\Gamma}^{\rho}_{[\lambda|\sigma}\mathbf{W}^{\tau\sigma}{}_{,\nu]} - \tilde{\Gamma}^{\sigma}_{\sigma[\lambda}\mathbf{W}^{\tau\rho}{}_{,\nu]} \qquad \text{(E.8)}\\
=&\; \tilde{\Gamma}^{\tau}_{\sigma[\lambda,\nu]}\mathbf{W}^{\sigma\rho} + \tilde{\Gamma}^{\rho}_{[\lambda|\sigma,|\nu]}\mathbf{W}^{\tau\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma[\lambda,\nu]}\mathbf{W}^{\tau\rho} - \frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\rho}{}_{,[\nu}\delta^{\tau}_{\lambda]} - \hat{\mathbf{j}}^{\tau}{}_{,[\nu}\delta^{\rho}_{\lambda]})\\
&-\tilde{\Gamma}^{\tau}_{\sigma[\lambda}\left(\tilde{\Gamma}^{\sigma}_{\alpha|\nu]}\mathbf{W}^{\alpha\rho} + \tilde{\Gamma}^{\rho}_{\nu]\alpha}\mathbf{W}^{\sigma\alpha} - \tilde{\Gamma}^{\alpha}_{\nu]\alpha}\mathbf{W}^{\sigma\rho} - \frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\rho}\delta^{\sigma}_{\nu]} - \hat{\mathbf{j}}^{\sigma}\delta^{\rho}_{\nu]})\right)\\
&-\tilde{\Gamma}^{\rho}_{[\lambda|\sigma}\left(\tilde{\Gamma}^{\tau}_{\alpha|\nu]}\mathbf{W}^{\alpha\sigma} + \tilde{\Gamma}^{\sigma}_{\nu]\alpha}\mathbf{W}^{\tau\alpha} - \tilde{\Gamma}^{\alpha}_{\nu]\alpha}\mathbf{W}^{\tau\sigma} - \frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\sigma}\delta^{\tau}_{\nu]} - \hat{\mathbf{j}}^{\tau}\delta^{\sigma}_{\nu]})\right)\\
&+\tilde{\Gamma}^{\sigma}_{\sigma[\lambda}\left(\tilde{\Gamma}^{\tau}_{\alpha|\nu]}\mathbf{W}^{\alpha\rho} + \tilde{\Gamma}^{\rho}_{\nu]\alpha}\mathbf{W}^{\tau\alpha} - \tilde{\Gamma}^{\alpha}_{\nu]\alpha}\mathbf{W}^{\tau\rho} - \frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\rho}\delta^{\tau}_{\nu]} - \hat{\mathbf{j}}^{\tau}\delta^{\rho}_{\nu]})\right). \qquad \text{(E.9)}
\end{aligned}$$

Cancelling the terms 2B-3A, 2C-4A, 3C-4B and using (E.3) gives,

$$\begin{aligned}
0 \;=&\; \frac{1}{2}\left[\mathbf{W}^{\sigma\rho}\tilde{\mathcal{R}}^{\tau}{}_{\sigma\nu\lambda} + \mathbf{W}^{\tau\sigma}\mathcal{R}^{\rho}{}_{\sigma\nu\lambda}(\tilde{\Gamma}^T)\right] + \frac{4\pi}{(n-1)}\left[\tilde{\Gamma}^{\tau}_{[\nu\lambda]}\hat{\mathbf{j}}^{\rho} - \tilde{\Gamma}^{\rho}_{[\lambda\nu]}\hat{\mathbf{j}}^{\tau}\right]\\
&+\frac{4\pi}{(n-1)}\left[(\hat{\mathbf{j}}^{\tau}{}_{,[\nu} + \tilde{\Gamma}^{\tau}_{\sigma[\nu}\hat{\mathbf{j}}^{\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma[\nu}\hat{\mathbf{j}}^{\tau})\delta^{\rho}_{\lambda]} - (\hat{\mathbf{j}}^{\rho}{}_{,[\nu} + \tilde{\Gamma}^{\rho}_{[\nu|\sigma}\hat{\mathbf{j}}^{\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma[\nu}\hat{\mathbf{j}}^{\rho})\delta^{\tau}_{\lambda]}\right]. \qquad \text{(E.10)}
\end{aligned}$$

Multiplying by 2, contracting over ${}^{\rho}_{\nu}$, and using (E.5) and $\hat{\mathbf{j}}^{\nu}_{,\nu}\!=\!0$ from (2.49) gives,

$$
\begin{aligned}
0 \;&=\; \mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}^{\tau}{}_{\sigma\nu\lambda} + \mathbf{W}^{\tau\sigma}\mathcal{R}^{\nu}{}_{\sigma\nu\lambda}(\tilde{\Gamma}^{T}) + \frac{8\pi}{(n-1)}\left[\tilde{\Gamma}^{\tau}_{[\nu\lambda]}\hat{\mathbf{j}}^{\nu} - \tilde{\Gamma}^{\nu}_{[\lambda\nu]}\hat{\mathbf{j}}^{\tau}\right] \\
&\quad + \frac{8\pi}{(n-1)}\left[(\hat{\mathbf{j}}^{\tau}{}_{,[\nu} + \tilde{\Gamma}^{\tau}_{\sigma[\nu}\hat{\mathbf{j}}^{\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma[\nu}\hat{\mathbf{j}}^{\tau})\delta^{\nu}_{\lambda]} - (\hat{\mathbf{j}}^{\nu}{}_{,[\nu} + \tilde{\Gamma}^{\nu}_{[\nu|\sigma}\hat{\mathbf{j}}^{\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma[\nu}\hat{\mathbf{j}}^{\nu})\delta^{\tau}_{\lambda]}\right] \quad \text{(E.11)} \\
&=\; \mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}^{\tau}{}_{\sigma\nu\lambda} + \mathbf{W}^{\tau\sigma}\tilde{\mathcal{R}}_{\lambda\sigma} - \frac{4\pi(n-2)}{(n-1)}(\hat{\mathbf{j}}^{\tau}{}_{,\lambda} + \tilde{\Gamma}^{\tau}_{\sigma\lambda}\hat{\mathbf{j}}^{\sigma} - \tilde{\Gamma}^{\sigma}_{\sigma\lambda}\hat{\mathbf{j}}^{\tau}). \quad \text{(E.12)}
\end{aligned}
$$

This is a generalization of the symmetry $R^{\tau}{}_{\lambda} = R_{\lambda}{}^{\tau}$ of the ordinary Ricci tensor.

Next we will use the generalized uncontracted Bianchi identity[3], which can be verified by direct computation,

$$
\tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\underset{+}{\sigma}\underset{-}{\nu}\underset{+}{\alpha};\lambda} + \tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\underset{+}{\sigma}\underset{+}{\alpha}\underset{+}{\lambda};\nu} + \tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\underset{+}{\sigma}\underset{-}{\lambda}\underset{-}{\nu};\alpha} = 0. \quad \text{(E.13)}
$$

The $+/-$ notation is from [3] and indicates that covariant derivative is being done with $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ instead of the usual $\Gamma^{\alpha}_{\nu\mu}$. A plus by an index means that the associated derivative index is to be placed on the right side of the connection, and a minus means that it is to be placed on the left side. Note that the identity (E.13) is true for either the ordinary Riemann tensor or for our definition (E.3). This is because the two tensors differ by the term $\delta^{\tau}_{\nu}\tilde{\Gamma}^{\sigma}_{\sigma[\alpha,\mu]}$, so that the expression (E.13) would differ by the term $\delta^{\tau}_{\sigma}(\tilde{\Gamma}^{\rho}_{\rho[\underset{-}{\nu},\underset{+}{\alpha}];\lambda} + \tilde{\Gamma}^{\rho}_{\rho[\underset{+}{\alpha},\underset{+}{\lambda}];\nu} + \tilde{\Gamma}^{\rho}_{\rho[\underset{-}{\lambda},\underset{-}{\nu}];\alpha})$. But this difference vanishes because for an arbitrary curl $Y_{[\alpha,\lambda]}$ we have

$$
\begin{aligned}
Y_{[\underset{-}{\nu},\underset{+}{\alpha}];\lambda} + Y_{[\underset{+}{\alpha},\underset{+}{\lambda}];\nu} + Y_{[\underset{-}{\lambda},\underset{-}{\nu}];\alpha} \;&=\; Y_{[\nu,\alpha],\lambda} - \tilde{\Gamma}^{\sigma}_{\lambda\nu}Y_{[\sigma,\alpha]} - \tilde{\Gamma}^{\sigma}_{\alpha\lambda}Y_{[\nu,\sigma]} \\
&\quad + \; Y_{[\alpha,\lambda],\nu} - \tilde{\Gamma}^{\sigma}_{\alpha\nu}Y_{[\sigma,\lambda]} - \tilde{\Gamma}^{\sigma}_{\lambda\nu}Y_{[\alpha,\sigma]} \\
&\quad + \; Y_{[\lambda,\nu],\alpha} - \tilde{\Gamma}^{\sigma}_{\alpha\lambda}Y_{[\sigma,\nu]} - \tilde{\Gamma}^{\sigma}_{\alpha\nu}Y_{[\lambda,\sigma]} = 0. \quad \text{(E.14)}
\end{aligned}
$$

A simple form of the generalized contracted Bianchi identity results if we contract

(E.13) over $\mathbf{W}^{\sigma\nu}$ and ${}^{\tau}_{\alpha}$, then substitute (E.12) for $\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}^{\tau}{}_{\sigma\nu\lambda}$ and (E.6) for $\mathbf{W}^{\overset{+}{\sigma}\overset{-}{\nu}}{}_{;\tau}$,

$$
\begin{aligned}
0 &= \mathbf{W}^{\sigma\nu}(\tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\underset{+}{\sigma}\underset{-}{\nu}\underset{+}{\tau};\lambda}+\tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\underset{+}{\sigma}\underset{+}{\tau}\underset{+}{\lambda};\nu}+\tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\underset{+}{\sigma}\underset{-}{\lambda}\underset{-}{\nu};\tau}) &\text{(E.15)}\\
&= -\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\underset{+}{\sigma}\underset{-}{\nu};\lambda}+\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\underset{+}{\sigma}\underset{+}{\lambda};\nu}-\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\underset{+}{\sigma}\underset{-}{\nu}\underset{-}{\lambda};\tau} &\text{(E.16)}\\
&= -\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\underset{+}{\sigma}\underset{-}{\nu};\lambda}+(\mathbf{W}^{\sigma\overset{-}{\nu}}\tilde{\mathcal{R}}_{\sigma\underset{+}{\lambda}})_{;\nu}-(\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}^{\overset{+}{\tau}}{}_{\sigma\nu\underset{-}{\lambda}})_{;\tau}\\
&\qquad -\mathbf{W}^{\overset{+}{\sigma}\overset{-}{\nu}}{}_{;\nu}\tilde{\mathcal{R}}_{\sigma\lambda}+\mathbf{W}^{\overset{+}{\sigma}\overset{-}{\nu}}{}_{;\tau}\tilde{\mathcal{R}}^{\tau}{}_{\sigma\nu\lambda} &\text{(E.17)}\\
&= -\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\underset{+}{\sigma}\underset{-}{\nu};\lambda}+(\mathbf{W}^{\sigma\overset{-}{\nu}}\tilde{\mathcal{R}}_{\sigma\underset{+}{\lambda}})_{;\nu}\\
&\quad +\left(\mathbf{W}^{\overset{+}{\tau}\sigma}\tilde{\mathcal{R}}_{\underset{-}{\lambda}\sigma}-\frac{4\pi(n-2)}{(n-1)}(\hat{\mathbf{j}}^{\overset{+}{\tau}}{}_{,\underset{-}{\lambda}}+\tilde{\Gamma}^{\overset{+}{\tau}}_{\sigma\underset{-}{\lambda}}\hat{\mathbf{j}}^{\sigma}-\tilde{\Gamma}^{\sigma}_{\sigma\underset{-}{\lambda}}\hat{\mathbf{j}}^{\overset{+}{\tau}})\right)_{;\tau}\\
&\quad -\frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\nu}\delta^{\sigma}_{\nu}-\hat{\mathbf{j}}^{\sigma}\delta^{\nu}_{\nu})\tilde{\mathcal{R}}_{\sigma\lambda}+\frac{4\pi}{(n-1)}(\hat{\mathbf{j}}^{\nu}\delta^{\sigma}_{\tau}-\hat{\mathbf{j}}^{\sigma}\delta^{\nu}_{\tau})\tilde{\mathcal{R}}^{\tau}{}_{\sigma\nu\lambda} &\text{(E.18)}\\
&= -\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\underset{+}{\sigma}\underset{-}{\nu};\lambda}+(\mathbf{W}^{\sigma\overset{-}{\nu}}\tilde{\mathcal{R}}_{\sigma\underset{+}{\lambda}})_{;\nu}+(\mathbf{W}^{\overset{+}{\nu}\sigma}\tilde{\mathcal{R}}_{\underset{-}{\lambda}\sigma})_{;\nu}\\
&\quad -\frac{4\pi(n-2)}{(n-1)}(\hat{\mathbf{j}}^{\overset{+}{\tau}}{}_{,\underset{-}{\lambda}}+\tilde{\Gamma}^{\overset{+}{\tau}}_{\sigma\underset{-}{\lambda}}\hat{\mathbf{j}}^{\sigma}-\tilde{\Gamma}^{\sigma}_{\sigma\underset{-}{\lambda}}\hat{\mathbf{j}}^{\overset{+}{\tau}})_{;\tau}\\
&\quad +\frac{4\pi(n-2)}{(n-1)}\hat{\mathbf{j}}^{\sigma}\tilde{\mathcal{R}}_{\sigma\lambda}+\frac{4\pi}{(n-1)}\hat{\mathbf{j}}^{\nu}\tilde{\mathcal{R}}^{\sigma}{}_{\sigma\nu\lambda} &\text{(E.19)}
\end{aligned}
$$

$$
\begin{aligned}
= \;& -\mathbf{W}^{\sigma\nu}(\tilde{\mathcal{R}}_{\sigma\nu,\lambda}-\tilde{\Gamma}^{\alpha}_{\sigma\lambda}\tilde{\mathcal{R}}_{\alpha\nu}-\tilde{\Gamma}^{\alpha}_{\lambda\nu}\tilde{\mathcal{R}}_{\sigma\alpha})\\
& +(\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\sigma\lambda})_{,\nu}+\tilde{\Gamma}^{\nu}_{\nu\alpha}\mathbf{W}^{\sigma\alpha}\tilde{\mathcal{R}}_{\sigma\lambda}-\tilde{\Gamma}^{\alpha}_{\lambda\nu}\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\sigma\alpha}-\tilde{\Gamma}^{\alpha}_{\alpha\nu}\mathbf{W}^{\sigma\nu}\tilde{\mathcal{R}}_{\sigma\lambda}\\
& +(\mathbf{W}^{\nu\sigma}\tilde{\mathcal{R}}_{\lambda\sigma})_{,\nu}+\tilde{\Gamma}^{\nu}_{\alpha\nu}\mathbf{W}^{\alpha\sigma}\tilde{\mathcal{R}}_{\lambda\sigma}-\tilde{\Gamma}^{\alpha}_{\nu\lambda}\mathbf{W}^{\nu\sigma}\tilde{\mathcal{R}}_{\alpha\sigma}-\tilde{\Gamma}^{\alpha}_{\alpha\nu}\mathbf{W}^{\nu\sigma}\tilde{\mathcal{R}}_{\lambda\sigma}\\
& -\frac{4\pi(n-2)}{(n-1)}[\hat{\mathbf{j}}^{\tau}{}_{,\lambda,\tau}+\tilde{\Gamma}^{\tau}_{\sigma\lambda,\tau}\hat{\mathbf{j}}^{\sigma}+\tilde{\Gamma}^{\tau}_{\sigma\lambda}\hat{\mathbf{j}}^{\sigma}_{,\tau}-\tilde{\Gamma}^{\sigma}_{\sigma\lambda,\tau}\hat{\mathbf{j}}^{\tau}-\tilde{\Gamma}^{\sigma}_{\sigma\lambda}\hat{\mathbf{j}}^{\tau}_{,\tau}\\
&\qquad +\tilde{\Gamma}^{\tau}_{\alpha\tau}(\hat{\mathbf{j}}^{\alpha}{}_{,\lambda}+\tilde{\Gamma}^{\alpha}_{\sigma\lambda}\hat{\mathbf{j}}^{\sigma}-\tilde{\Gamma}^{\sigma}_{\sigma\lambda}\hat{\mathbf{j}}^{\alpha})\\
&\qquad -\tilde{\Gamma}^{\alpha}_{\tau\lambda}(\hat{\mathbf{j}}^{\tau}{}_{,\alpha}+\tilde{\Gamma}^{\tau}_{\sigma\alpha}\hat{\mathbf{j}}^{\sigma}-\tilde{\Gamma}^{\sigma}_{\sigma\alpha}\hat{\mathbf{j}}^{\tau})\\
&\qquad -\tilde{\Gamma}^{\alpha}_{\alpha\tau}(\hat{\mathbf{j}}^{\tau}{}_{,\lambda}+\tilde{\Gamma}^{\tau}_{\sigma\lambda}\hat{\mathbf{j}}^{\sigma}-\tilde{\Gamma}^{\sigma}_{\sigma\lambda}\hat{\mathbf{j}}^{\tau})\\
&\qquad -\hat{\mathbf{j}}^{\sigma}(\tilde{\mathcal{R}}_{\sigma\lambda}-\tilde{\Gamma}^{\alpha}_{\alpha[\sigma,\lambda]})-\hat{\mathbf{j}}^{\sigma}(\tilde{\Gamma}^{\alpha}_{\alpha\sigma,\lambda}-\tilde{\Gamma}^{\alpha}_{\alpha\lambda,\sigma})] \qquad \text{(E.20)}
\end{aligned}
$$

With the $\hat{\mathbf{j}}^{\sigma}$ terms of (E.20), 4C-6A,4D-8D,5A-7A,5B-7B,5C-7C all cancel, 4A and

4E are zero because $\hat{\mathbf{j}}^{\nu}_{,\nu}=0$ from (2.49), and 4B,6B,6C,8C cancel the Ricci tensor term 8A,8B. With the $\mathbf{W}^{\tau\sigma}$ terms of (E.20), all those with a $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ factor cancel, which are the terms 1C-2C,1B-3C,2B-2D,3B-3D. Doing the cancellations and using (E.1) we get

$$0 \;=\; (\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\lambda}+\sqrt{-N}N^{\dashv\sigma\nu}\tilde{\mathcal{R}}_{\lambda\sigma})_{,\nu}-\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\nu,\lambda}. \tag{E.21}$$

Equation (E.21) is a simple generalization of the ordinary contracted Bianchi identity $2(\sqrt{-g}\,R^{\nu}{}_{\lambda})_{,\nu}-\sqrt{-g}\,g^{\nu\sigma}R_{\sigma\nu,\lambda}=0$, and it applies even when $j^{\tau}\neq 0$. Because $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ has cancelled out of (E.21), the Christoffel connection $\Gamma^{\alpha}_{\nu\mu}$ would also cancel, so a manifestly tensor relation can be obtained by replacing the ordinary derivatives with covariant derivatives done with $\Gamma^{\alpha}_{\nu\mu}$,

$$0 \;=\; (\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\lambda}+\sqrt{-N}N^{\dashv\sigma\nu}\tilde{\mathcal{R}}_{\lambda\sigma})_{;\nu}-\sqrt{-N}N^{\dashv\nu\sigma}\tilde{\mathcal{R}}_{\sigma\nu;\lambda}. \tag{E.22}$$

Rewriting the identity in terms of $g^{\rho\tau}$ and $\hat{f}^{\rho\tau}$ as defined by (E.1,E.2) gives,

$$0 \;=\; (\sqrt{-g}\,(g^{\sigma\nu}+\hat{f}^{\sigma\nu})\tilde{\mathcal{R}}_{\sigma\lambda}+\sqrt{-g}\,(g^{\nu\sigma}+\hat{f}^{\nu\sigma})\tilde{\mathcal{R}}_{\lambda\sigma})_{;\nu}-\sqrt{-g}\,(g^{\sigma\nu}+\hat{f}^{\sigma\nu})\tilde{\mathcal{R}}_{\sigma\nu;\lambda} \tag{E.23}$$

$$=\; \sqrt{-g}\,[2\tilde{\mathcal{R}}^{(\nu}{}_{\lambda);\nu}-\tilde{\mathcal{R}}^{\sigma}_{\sigma;\lambda}]+\sqrt{-g}\,[2(\hat{f}^{\nu\sigma}\tilde{\mathcal{R}}_{[\lambda\sigma]})_{;\nu}+\hat{f}^{\nu\sigma}\tilde{\mathcal{R}}_{[\sigma\nu];\lambda}] \tag{E.24}$$

$$=\; \sqrt{-g}\,[2\tilde{\mathcal{R}}^{(\nu}{}_{\lambda);\nu}-\tilde{\mathcal{R}}^{\sigma}_{\sigma;\lambda}]+\sqrt{-g}\,[3\hat{f}^{\nu\sigma}\tilde{\mathcal{R}}_{[\sigma\nu,\lambda]}+2\hat{f}^{\nu\sigma}{}_{;\nu}\tilde{\mathcal{R}}_{[\lambda\sigma]}]. \tag{E.25}$$

Dividing by $2\sqrt{-g}$ gives another form of the generalized contracted Bianchi identity

$$\left(\tilde{\mathcal{R}}^{(\nu}{}_{\lambda)}-\frac{1}{2}\delta^{\nu}_{\lambda}\tilde{\mathcal{R}}^{\sigma}_{\sigma}\right)_{;\nu}=\frac{3}{2}\hat{f}^{\nu\sigma}\tilde{\mathcal{R}}_{[\nu\sigma,\lambda]}+\hat{f}^{\nu\sigma}{}_{;\nu}\tilde{\mathcal{R}}_{[\sigma\lambda]}. \tag{E.26}$$

From (E.2,2.40) we get the final result (4.3).

Appendix F

Validation of the EIH method to post-Coulombian order

Here we state the post-Coulombian equations of motion of Einstein-Maxwell theory obtained by two authors[72, 74] using the EIH method, and show that they match the equations of motion obtained from the Darwin Lagrangian[53]. For two particles the Darwin Lagrangian takes the form

$$L_a = \frac{m_a v_a^2}{2} + \frac{1}{8}\frac{m_a v_a^4}{c^2} - e_a \frac{e_b}{R_{ab}} + \frac{e_a}{2c^2}\frac{e_b}{R_{ab}}\left[\mathbf{v}_a \cdot \mathbf{v}_b + (\mathbf{v}_a \cdot \mathbf{n}_{ab})(\mathbf{v}_b \cdot \mathbf{n}_{ab})\right]. \tag{F.1}$$

Here we are using the notation

$$\dot{r}_a^i = v_a^i, \quad \dot{r}_b^i = v_b^i, \quad r_{ab}^i = r_a^i - r_b^i, \quad v_{ab}^i = v_a^i - v_b^i, \quad n_{ab}^i = r_{ab}^i / R_{ab}, \quad R_{ab}^2 = r_{ab}^i r_{ab}^i. \tag{F.2}$$

From this we get the equations of motion

$$
\begin{aligned}
0 &= \frac{\partial L_a}{\partial r_a^i} - \frac{\partial}{\partial t}\left(\frac{\partial L_a}{\partial v_a^i}\right) \qquad \text{(F.3)}\\
&= e_b e_b \frac{r_{ab}^i}{R_{ab}^3} + \frac{e_a e_b}{2c^2}\left(-\frac{r_{ab}^i}{R_{ab}^3} v_a^s v_b^s - \frac{3r_{ab}^i}{R_{ab}^5} v_a^s r_{ab}^s v_b^u r_{ab}^u + \frac{v_a^i}{R_{ab}^3} v_b^s r_{ab}^s + \frac{v_b^i}{R_{ab}^3} v_a^s r_{ab}^s\right)\\
&\quad -m_a \dot{v}_a^i - \frac{m_a}{2c^2}(\dot{v}_a^i v_a^2 + 2v_a^i v_a^s \dot{v}_a^s) - \frac{e_a e_b}{2c^2 R_{ab}}\left(\dot{v}_b^i - v_b^i \frac{v_{ab}^i r_{ab}^s}{R_{ab}^2}\right)\\
&\quad -\frac{e_a e_b}{2c^2 R_{ab}^3}\left(v_{ab}^i v_b^s r_{ab}^s + r_{ab}^i \dot{v}_b^s r_{ab}^s + r_{ab}^i v_b^s v_{ab}^s - 3r_{ab}^i v_b^u r_{ab}^u \frac{v_{ab}^s r_{ab}^s}{R_{ab}^2}\right) \qquad \text{(F.4)}\\
&= -m\dot{v}_a^i + e_a e_b \frac{r_{ab}^i}{R_{ab}^3} + \frac{e_a e_b}{c^2}\left[-\frac{v_a^2}{2} - v_a^s v_b^s + \frac{v_b^2}{2}\right]\frac{r_{ab}^i}{R_{ab}^3}\\
&\quad +\frac{e_a e_b}{c^2}\left[-v_a^s v_a^i + v_a^s v_b^i\right]\frac{r_{ab}^s}{R_{ab}^3} - \frac{3e_a e_b}{2c^2} v_b^u v_b^s \frac{r_{ab}^u r_{ab}^s r_{ab}^i}{R_{ab}^5} + \frac{e_a^2 e_b^2}{m_b c^2}\frac{r_{ab}^i}{R_{ab}^4}. \qquad \text{(F.5)}
\end{aligned}
$$

Let us first compare the notation used in the various references,

Landau/Lifshitz	r_a^i	r_b^i	r_{ab}^i	R_{ab}	e_a	e_b	m_a	m_b
Wallace	η^i	ζ^i	β_i	r	e_1	e_2	m_1	m_2
Gorbatenko	ξ^i	η^i	$-R_i$	R	Q	q	M	m
Bazanski	ξ^i	η^i	$-R_i$	r	e_1	e_2	m_1	m_2
Anderson	x_A^i	x_B^i	x_{AB}^i	x_{AB}	q_A	q_B	m_A	m_B
Jackson	r_1^i	r_2^i	r_{12}^i	R	q_1	q_2	m_1	m_2

(F.6)

The Wallace[72] equations of motion (including radiation reaction term) are

$$
\begin{aligned}
m_1\ddot{\eta}^m + e_1 e_2 \frac{\partial}{\partial \eta^m}\left(\frac{1}{r}\right) &= e_1 e_2 \left[\left(\frac{1}{2}\dot{\eta}^s\dot{\eta}^s + \dot{\eta}^s\dot{\zeta}^s\right)\frac{\partial}{\partial \eta^m}\left(\frac{1}{r}\right)\right.\\
&\quad \left. + (\dot{\eta}^s\dot{\eta}^m - \dot{\eta}^s\dot{\zeta}^m + \dot{\zeta}^s\dot{\zeta}^m)\frac{\partial}{\partial \eta^s}\left(\frac{1}{r}\right) - \frac{1}{2}\frac{\partial^3 r}{\partial \eta^m \eta^r \eta^s}\dot{\zeta}^r\dot{\zeta}^s\right]\\
&\quad - \frac{e_1^2 e_2^2}{m_2}\frac{1}{r}\frac{\partial}{\partial \eta^m}\left(\frac{1}{r}\right) + \frac{2}{3}e_1(e_1\dddot{\eta}^m + e_2\dddot{\zeta}^m). \qquad \text{(F.7)}
\end{aligned}
$$

Using

$$
\frac{\partial}{\partial \eta^m}\left(\frac{1}{r}\right) = -\frac{\beta_m}{r^3}, \quad \frac{\partial r}{\partial \eta^s} = \frac{1}{r}\beta_s, \quad \frac{\partial^2 r}{\partial \eta^r \eta^s} = -\frac{\beta_r\beta_s}{r^3} + \frac{1}{r}\delta_{sr} \qquad \text{(F.8)}
$$

$$
\frac{\partial^3 r}{\partial \eta^m \eta^r \eta^s} = -\delta_{rm}\frac{\beta_s}{r^3} - \delta_{sm}\frac{\beta_r}{r^3} + \frac{3\beta_r\beta_s\beta_m}{r^5} - \frac{\beta_m}{r^3}\delta_{sr} \qquad \text{(F.9)}
$$

$$
-\frac{1}{2}\frac{\partial^3 r}{\partial \eta^m \eta^r \eta^s}\dot{\zeta}^r\dot{\zeta}^s = \frac{\dot{\zeta}^m\dot{\zeta}^s\beta_s}{r^3} - \frac{3\dot{\zeta}^r\dot{\zeta}^s\beta_r\beta_s\beta_m}{2r^5} + \frac{\beta_m\dot{\zeta}^s\dot{\zeta}^s}{2r^3}, \qquad \text{(F.10)}
$$

we get

$$m_1\ddot{\eta}^m - e_1e_2\frac{\beta_m}{r^3} = e_1e_2\left[-\left(\frac{1}{2}\dot{\eta}^s\dot{\eta}^s + \dot{\eta}^s\dot{\zeta}^s\right)\frac{\beta_m}{r^3}\right.$$
$$\left. - (\dot{\eta}^s\dot{\eta}^m - \dot{\eta}^s\dot{\zeta}^m + \dot{\zeta}^s\dot{\zeta}^m)\frac{\beta_s}{r^3} - \frac{1}{2}\frac{\partial^3 r}{\partial\eta^m\eta^r\eta^s}\dot{\zeta}^r\dot{\zeta}^s\right]$$
$$+ \frac{e_1^2e_2^2}{m_2}\frac{1}{r}\frac{\beta_m}{r^3} + \frac{2}{3}e_1(e_1\dddot{\eta}^m + e_2\dddot{\zeta}^m) \tag{F.11}$$
$$= e_1e_2\left[-\left(\frac{1}{2}\dot{\eta}^s\dot{\eta}^s + \dot{\eta}^s\dot{\zeta}^s - \frac{\dot{\zeta}^s\dot{\zeta}^s}{2}\right)\frac{\beta_m}{r^3}\right.$$
$$\left. - (\dot{\eta}^s\dot{\eta}^m - \dot{\eta}^s\dot{\zeta}^m)\frac{\beta_s}{r^3} - \frac{3\dot{\zeta}^r\dot{\zeta}^s\beta_r\beta_s\beta_m}{2r^5}\right]$$
$$+ \frac{e_1^2e_2^2}{m_2}\frac{\beta_m}{r^4} + \frac{2}{3}e_1(e_1\dddot{\eta}^m + e_2\dddot{\zeta}^m). \tag{F.12}$$

Translating this into the Landau/Lifshitz notation we see that it agrees with (F.5),

$$m_a\dot{v}^m - e_ae_b\frac{r_{ab}^m}{R_{ab}^3} = e_ae_b\left[-\left(\frac{v_a^2}{2} + v_a^sv_b^s - \frac{v_b^2}{2}\right)\frac{r_{ab}^m}{R_{ab}^3}\right.$$
$$\left. - (v_a^sv_a^m - v_a^sv_b^m)\frac{r_{ab}^s}{R_{ab}^3} - \frac{3v_b^rv_b^sr_{ab}^rr_{ab}^sr_{ab}^m}{2R_{ab}^5}\right]$$
$$+ \frac{e_a^2e_b^2}{m_b}\frac{r_{ab}^m}{R_{ab}^4} + \frac{2}{3}e_a(e_a\ddot{v}_a^m + e_b\ddot{v}_b^m). \tag{F.13}$$

The Gorbatenko[74] equations of motion (including radiation reaction term) are

$$M\ddot{\xi}_k = -\frac{qQ}{R^3}R_k + qQ\left[\frac{(\dot{\xi}_l\dot{\eta}_l)}{R^3}R_k - \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\eta}_k + \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\xi}_k + \frac{(\dot{\xi}_l\dot{\xi}_l)}{2R^3}R_k\right.$$
$$\left. -\frac{\ddot{\eta}_k}{2R} - \frac{(R_l\ddot{\eta}_l)}{2R^3}R_k - \frac{3}{2}\frac{(R_l\dot{\eta}_l)^2}{R^5}R_k - \frac{(\dot{\eta}_l\dot{\eta}_l)}{2R^3}R_k\right] + \frac{2}{3}(Q\dddot{\xi}_k + q\dddot{\eta}_k)Q. \tag{F.14}$$

The Coulombian order equations for the η^k particle are the first two terms but with $\xi^k \to \eta^k$,$M \to m$, $Q \to q$, $q \to Q$, $R_k \to -R_k$. Using these equations we have

$$m\ddot{\eta}_k \approx \frac{qQ}{R^3}R_k \quad \Rightarrow mR_l\ddot{\eta}_l \approx \frac{qQ}{R} \quad \Rightarrow -\frac{(R_l\ddot{\eta}_l)}{2R^3}R_k \approx -\frac{qQ}{2m}\frac{R_k}{R^4} \tag{F.15}$$
$$\Rightarrow -\frac{\ddot{\eta}_k}{2R} - \frac{(R_l\ddot{\eta}_l)}{2R^3}R_k \approx -\frac{qQR_k}{mR^4}. \tag{F.16}$$

Substituting this last equation into (F.14) and assuming $1/(mR)$ is $\mathcal{O}(\lambda^1)$ gives

$$\begin{aligned} M\ddot{\xi}_k &= -\frac{qQ}{R^3}R_k + qQ\left[\frac{(\dot{\xi}_l\dot{\eta}_l)}{R^3}R_k - \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\eta}_k + \frac{(R_l\dot{\xi}_l)}{R^3}\dot{\xi}_k + \frac{(\dot{\xi}_l\dot{\xi}_l)}{2R^3}R_k\right. \\ &\qquad \left. -\frac{qQR_k}{mR^4} - \frac{3}{2}\frac{(R_l\dot{\eta}_l)^2}{R^5}R_k - \frac{(\dot{\eta}_l\dot{\eta}_l)}{2R^3}R_k\right] + \frac{2}{3}(Q\,\dddot{\xi}_k + q\,\dddot{\eta}_k)Q. \quad \text{(F.17)} \end{aligned}$$

Translating this into the Landau/Lifshifz notation we see that it agrees with (F.5),

$$\begin{aligned} m_a\dot{v}_a^k &= \frac{e_b e_a}{R_{ab}^3}r_{ab}^k + e_b e_a\left[-\frac{v_a^l v_b^l}{R_{ab}^3}r_{ab}^k + \frac{r_{ab}^l v_a^l}{R_{ab}^3}v_b^k - \frac{r_{ab}^l v_a^l}{R_{ab}^3}v_a^k - \frac{v_a^2}{2R_{ab}^3}r_{ab}^k\right. \\ &\qquad \left. +\frac{e_b e_a r_{ab}^k}{m_b R_{ab}^4} - \frac{3}{2}\frac{(r_{ab}^l v_b^l)^2}{R_{ab}^5}r_{ab}^k + \frac{v_b^2}{2R_{ab}^3}r_{ab}^k\right] + \frac{2}{3}(e_a\ddot{v}_a^k + e_b\ddot{v}_b^k)e_a. \quad \text{(F.18)} \end{aligned}$$

Appendix G

Application of point-particle post-Newtonian methods

Here we apply point-particle post-Newtonian methods to LRES theory in order to calculate what the theory predicts for the Kreuzer experiment[86]. The Kreuzer experiment is an experiment which can distinguish between active gravitational mass and inertial mass. Active gravitational mass is the mass which is the source of the Newtonian gravitational potential m_A/r. Inertial mass is the mass which relates the acceleration of a body to an applied force. In particular inertial mass is the mass in the Lorentz force equation $mu^\nu u_{\mu;\nu} = -Q(A_{\nu,\mu} - A_{\mu,\nu})u^\nu$, which is exactly the same in our theory (4.10,4.11) as in Einstein-Maxwell theory.

In [86] the computations only require the lowest-order post-Newtonian version of the Lorentz force equation,

$$\frac{m_p dv_p^i}{dt} = m_p U_i^* + Q_p E_i \tag{G.1}$$

where

$$U^* = \sum_p \frac{m_p}{r_p} \tag{G.2}$$

$$E_i = A_{0,i}. \tag{G.3}$$

In [86] the computations also only require the lowest-order post-Newtonian approximation of the electromagnetic field,

$$f_{i0} = E_i = -\nabla_i \psi, \quad f_{ik} = 0, \tag{G.4}$$

where

$$\psi = \sum_p \frac{Q_p}{r_p}. \tag{G.5}$$

Here we will be using the notation

$$\mathbf{x} = (\textit{position of observer}), \quad \mathbf{x}_p = (\textit{position of particle } p) \tag{G.6}$$

$$\mathbf{r}_p = \mathbf{x} - \mathbf{x}_p, \quad r_p = |\mathbf{r}_p|, \quad \mathbf{r}_{pq} = \mathbf{x}_p - \mathbf{x}_q, \quad r_{pq} = |\mathbf{r}_{pq}|. \tag{G.7}$$

From our electric monopole solution (3.1,3.2,3.8) we have

$$f_{01} = \frac{Q}{r^2}\left(1 - \frac{2Q^2}{\Lambda_b r^4}\right)^{-1/2} \approx \frac{Q}{r^2}\left(1 + \frac{Q^2}{\Lambda_b r^4}\right), \tag{G.8}$$

$$F_{01} \approx \frac{Q}{r^2}\left(1 + \frac{4m}{\Lambda_b r^3} - \frac{4Q^2}{\Lambda_b r^4}\right). \tag{G.9}$$

The extra terms in (G.8,G.9) fall off as $1/r^5$ or $1/r^6$, and they all include a factor of $1/\Lambda_b$ and are $< 10^{-66}$ of the Q/r^2 term for worst-case radii accessible to measurement. Based upon this result and the close approximation of equations (2.47,2.48) to the ordinary Maxwell equations, we will also assume the approximation (G.4). Therefore we have $f_{[\nu\beta;\alpha]} = 0$, $f_{\nu\mu;}{}^{\nu} = 0$, and the 00 component of our effective electromagnetic energy momentum tensor (2.68,2.67) is

$$
\begin{aligned}
8\pi\tilde{T}^{EM}_{00} &\approx 2\left(f_0{}^{\nu} f_{\nu 0} - \frac{1}{4} g_{00} f^{\rho\nu} f_{\nu\rho}\right)\\
&+ \left(2f^{\tau}{}_{(0}f_{0)}{}^{\alpha}{}_{;\tau;\alpha} + 2f^{\alpha\tau} f_{\tau(0;\,0);\alpha} - f^{\nu}{}_{0;\alpha} f^{\alpha}{}_{0;\nu} + f^{\nu}{}_{0;\alpha} f_{\nu 0;}{}^{\alpha} + \frac{1}{2} f^{\nu}{}_{\alpha;0} f^{\alpha}{}_{\nu;\,0}\right.\\
&\qquad \left. - g_{00} f^{\tau\beta} f_{\beta}{}^{\alpha}{}_{;\tau;\alpha} - \frac{1}{4}(f^{\rho\nu} f_{\nu\rho}),{}^{\alpha}{}_{;\alpha} g_{00} - \frac{3}{4} g_{00} f_{[\nu\beta;\alpha]} f^{[\nu\beta}{}_{;}{}^{\alpha]}\right)\Lambda_b^{-1} \qquad \text{(G.10)}\\
&\approx 2\left(f_0{}^{i} f_{i0} - \frac{1}{2} f^{0i} f_{i0}\right)\\
&+ \left(2f^{0i} f_{i(0;\,0);0} - f^{k}{}_{0;i} f^{i}{}_{0;k} + f^{k}{}_{0;\alpha} f_{k0;}{}^{\alpha} + f^{0}{}_{i;0} f^{i}{}_{0;\,0} - \frac{1}{2}(f^{0i} f_{i0}),{}^{\alpha}{}_{;\alpha}\right)\Lambda_b^{-1}. \qquad \text{(G.11)}
\end{aligned}
$$

Time derivatives result in higher order post-Newtonian terms, raising and lowering with $\eta_{\mu\nu}$ differs from $g_{\mu\nu}$ only by higher order post-Newtonian terms, and covariant derivative differs from ordinary derivative only by higher order post-Newtonian terms. Therefore we have

$$8\pi\tilde{T}^{EM}_{00} \approx +2\left(E^2 - \frac{1}{2}E^2\right) + \left(\frac{1}{2}(E^2)_{,k,k} - E_{k,i}E_{i,k} + E_{k,i}E_{k,i}\right)\Lambda_b^{-1}. \qquad \text{(G.12)}$$

From (G.4), the last two terms cancel and we have

$$8\pi\tilde{T}^{EM}_{00} \approx E^2 + \frac{1}{2\Lambda_b}\nabla^2(E^2) = |\nabla\psi|^2 + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2. \qquad \text{(G.13)}$$

Using $\nabla^2\psi = -4\pi\sum_p Q_p\,\delta(\mathbf{r}_p)$ from (G.5) we get the identity

$$\frac{1}{2}\nabla^2\psi^2 = \frac{1}{2}\nabla\cdot\nabla(\psi^2) = \nabla\cdot(\psi\nabla\psi) = |\nabla\psi|^2 - 4\pi\sum_p Q_p\,\delta(\mathbf{r}_p)\psi. \qquad \text{(G.14)}$$

Then we have

$$8\pi\tilde{T}_{00}^{EM} = \frac{1}{2}\nabla^2\psi^2 + 4\pi\sum_p Q_p\,\delta(\mathbf{r}_p)\sum_q \frac{Q_q}{r_q} + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2 \tag{G.15}$$

$$= \frac{1}{2}\nabla^2\psi^2 + 4\pi\sum_p\sum_q Q_p\,\delta(\mathbf{r}_p)\frac{Q_q}{r_q} + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2 \tag{G.16}$$

$$= \frac{1}{2}\nabla^2\psi^2 + 4\pi\sum_p Q_p\,\delta(\mathbf{r}_p)\sum_{q\neq p} \frac{Q_q}{r_{pq}} + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2 \tag{G.17}$$

$$= \nabla^2\left(\frac{1}{2}\psi^2 - \sum_p \frac{Q_p}{r_p}\sum_{q\neq p}\frac{Q_q}{r_{pq}}\right) + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2. \tag{G.18}$$

Using $\nabla^2 U^* = -4\pi\sum_p m_p\,\delta(\mathbf{r}_p)$ from (G.2) and including the mass part of $\tilde{T}_{00}$ we get

$$8\pi\tilde{T}_{00} = 4\pi\sum_p m_p\,\delta(\mathbf{r}_p) + \nabla^2\left(\frac{1}{2}\psi^2 - \sum_p \frac{Q_p}{r_p}\sum_{q\neq p}\frac{Q_q}{r_{pq}}\right) + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2 \tag{G.19}$$

$$= -\nabla^2 U^* + \nabla^2\left(\frac{1}{2}\psi^2 - \sum_p \frac{Q_p}{r_p}\sum_{q\neq p}\frac{Q_q}{r_{pq}}\right) + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2. \tag{G.20}$$

From [86], $G_{00} = \nabla^2 g_{00}/2$ to lowest post-Newtonian order, so the Einstein equations are

$$G_{00} = \frac{1}{2}\nabla^2 g_{00} = -\nabla^2 U^* + \nabla^2\left(\frac{1}{2}\psi^2 - \sum_p \frac{Q_p}{r_p}\sum_{q\neq p}\frac{Q_q}{r_{pq}}\right) + \frac{1}{2\Lambda_b}\nabla^2|\nabla\psi|^2. \tag{G.21}$$

This has the solution

$$g_{00} = 1 - 2U^* + \psi^2 - 2\sum_p \frac{Q_p}{r_p}\sum_{q\neq p}\frac{Q_q}{r_{pq}} + \frac{1}{\Lambda_b}|\nabla\psi|^2. \tag{G.22}$$

Using (G.2,G.5) we get

$$g_{00} = 1 - 2\sum_p \frac{m_p}{r_p} + \left(\sum_p \frac{Q_p}{r_p}\right)^2 - 2\sum_p \frac{Q_p}{r_p}\sum_{q\neq p}\frac{Q_q}{r_{pq}} + \frac{1}{\Lambda_b}\left|\sum_p \frac{Q_p r_p^i}{r_p^3}\right|^2. \tag{G.23}$$

The only difference between this expression and that of ordinary Einstein-Maxwell theory is the last term, and this term falls off as $1/r^4$. Since the difference between gravitational mass and inertial mass in [86] depends only on terms which fall off as $1/r$, these two masses are the same for our theory as for Einstein-Maxwell theory.

Appendix H

Alternative derivation of the Lorentz force equation

Here we check the results in §4.1 by deriving the Lorentz force equation (4.10) in a different way, using the field equations (2.28) and a simple form of the generalized contracted Bianchi identity (4.1). Let us make the definitions

$$\mathbf{W}^{\nu\sigma}=\sqrt{-N}N^{\dashv\sigma\nu}=\sqrt{-g}\,(g^{\nu\sigma}+\hat{f}^{\nu\sigma}),\quad \hat{f}^{\nu\mu}=f^{\nu\mu}\sqrt{2}\,i\Lambda_b^{-1/2},\quad \hat{j}^{\nu}=j^{\nu}\sqrt{2}\,i\Lambda_b^{-1/2}. \quad \text{(H.1)}$$

The generalized contracted Bianchi identity (4.1) then becomes

$$\begin{aligned} 0 &= (\mathbf{W}^{\nu\sigma}\,\tilde{\mathcal{R}}_{\nu\lambda}+\mathbf{W}^{\sigma\nu}\,\tilde{\mathcal{R}}_{\lambda\nu})_{;\sigma}-\mathbf{W}^{\nu\sigma}\,\tilde{\mathcal{R}}_{\nu\sigma;\lambda} &\text{(H.2)}\\ &= (\mathbf{W}^{\nu\sigma}\,\tilde{\mathcal{R}}_{\nu\lambda}+\mathbf{W}^{\sigma\nu}\,\tilde{\mathcal{R}}_{\lambda\nu})_{,\sigma}-\mathbf{W}^{\nu\sigma}\,\tilde{\mathcal{R}}_{\nu\sigma,\lambda}. &\text{(H.3)}\end{aligned}$$

From (2.56,2.22,2.4) we have

$$\sqrt{-N}N^{\dashv\rho\tau}{}_{;\nu}N_{\tau\rho} = \sqrt{-N}(N^{\dashv\rho\tau}{}_{,\nu}+\Gamma^{\rho}_{\alpha\nu}N^{\dashv\alpha\tau}+\Gamma^{\tau}_{\alpha\nu}N^{\dashv\rho\alpha})N_{\tau\rho} \quad (H.4)$$

$$= \sqrt{-N}(N^{\dashv\rho\tau}{}_{,\nu}N_{\tau\rho}+2\Gamma^{\alpha}_{\alpha\nu}) = -2(\sqrt{-N}\,)_{,\nu}+2\sqrt{-N}\,\Gamma^{\alpha}_{\alpha\nu} \quad (H.5)$$

$$= -2(\sqrt{-N}\,)_{;\nu}, \quad (H.6)$$

$$\sqrt{-g}\hat{f}^{\tau\rho}{}_{;\nu}N_{[\tau\rho]} = \sqrt{-g}\hat{f}^{\tau\rho}{}_{;\nu}N_{\tau\rho} = (\sqrt{-N}N^{\dashv\rho\tau})_{;\nu}N_{\tau\rho} \quad (H.7)$$

$$= \sqrt{-N}N^{\dashv\rho\tau}{}_{;\nu}N_{\tau\rho}+n(\sqrt{-N}\,)_{;\nu} \quad (H.8)$$

$$= (n-2)(\sqrt{-N}\,)_{;\nu}. \quad (H.9)$$

Making linear combinations of the field equations (2.28) gives,

$$\begin{aligned} 0 &= \sqrt{-N}N^{\dashv\sigma\nu}\left(\tilde{\mathcal{R}}_{\nu\lambda}+2A_{[\nu,\lambda]}\sqrt{2}\,i\Lambda_b^{1/2}+\Lambda_bN_{\nu\lambda}+\Lambda_eg_{\nu\lambda}-8\pi S_{\nu\lambda}\right)\\ &+ \sqrt{-N}N^{\dashv\nu\sigma}\left(\tilde{\mathcal{R}}_{\lambda\nu}+2A_{[\lambda,\nu]}\sqrt{2}\,i\Lambda_b^{1/2}+\Lambda_bN_{\lambda\nu}+\Lambda_eg_{\lambda\nu}-8\pi S_{\lambda\nu}\right)\\ &- \sqrt{-N}N^{\dashv\alpha\nu}\left(\tilde{\mathcal{R}}_{\nu\alpha}+2A_{[\nu,\alpha]}\sqrt{2}\,i\Lambda_b^{1/2}+\Lambda_bN_{\nu\alpha}+\Lambda_eg_{\nu\alpha}-8\pi S_{\nu\alpha}\right)\delta^\sigma_\lambda \quad (H.10)\\ &= \mathbf{W}^{\nu\sigma}\,\tilde{\mathcal{R}}_{\nu\lambda}+\mathbf{W}^{\sigma\nu}\,\tilde{\mathcal{R}}_{\lambda\nu}-\mathbf{W}^{\nu\alpha}\,\tilde{\mathcal{R}}_{\nu\alpha}\delta^\sigma_\lambda-16\pi\sqrt{-g}\,T^\sigma_\lambda\\ &- 4\sqrt{-g}(2f^{\nu\sigma}A_{[\nu,\lambda]}+f^{\alpha\nu}A_{[\nu,\alpha]}\delta^\sigma_\lambda)+\Lambda_b\sqrt{-N}\,(2-n)\delta^\sigma_\lambda+\Lambda_e\sqrt{-g}\,(2-n)\delta^\sigma_\lambda. \quad (H.11) \end{aligned}$$

Using (H.2,H.9) the divergence of this equation gives the Lorentz force equation,

$$\begin{aligned} 0 &= \sqrt{-g}\,\hat{f}^{\alpha\nu}{}_{;\lambda}\tilde{\mathcal{R}}_{\nu\alpha}-16\pi\sqrt{-g}\,T^\sigma_{\lambda;\sigma}-\sqrt{-g}\Lambda_b\hat{f}^{\tau\rho}{}_{;\lambda}N_{[\tau\rho]}\\ &- 4\sqrt{-g}\left(-8\pi j^\nu A_{[\nu,\lambda]}+2f^{\nu\sigma}A_{[\nu,\lambda];\sigma}+f^{\alpha\nu}{}_{;\lambda}A_{[\nu,\alpha]}+f^{\alpha\nu}A_{[\nu,\alpha];\lambda}\right) \quad (H.12)\\ &= \sqrt{-g}\,\hat{f}^{\alpha\nu}{}_{;\lambda}(\tilde{\mathcal{R}}_{[\nu\alpha]}+2A_{[\nu,\alpha]}\sqrt{2}\,i\Lambda_b^{1/2}+\Lambda_bN_{[\nu\alpha]})-16\pi\sqrt{-g}\,T^\sigma_{\lambda;\sigma}\\ &- 4\sqrt{-g}\left(-8\pi j^\nu A_{[\nu,\lambda]}+3f^{\alpha\nu}A_{[\nu,\alpha,\lambda]}\right) \quad (H.13)\\ &= -16\pi\sqrt{-g}\left(T^\sigma_{\lambda;\sigma}+2j^\nu A_{[\lambda,\nu]}\right). \quad (H.14) \end{aligned}$$

Contracting H.11 and dividing by $(2-n)$ gives

$$0 = \mathbf{W}^{\nu\alpha}\,\tilde{\mathcal{R}}_{\nu\alpha} - 16\pi\sqrt{-g}\,\frac{1}{(2-n)}T^\sigma_\sigma + 4\sqrt{-g}\,f^{\alpha\nu}A_{[\nu,\alpha]} + \Lambda_b\sqrt{-N}\,n + \Lambda_e\sqrt{-g}\,n. \tag{H.15}$$

Adding this to H.11 gives

$$0 = \mathbf{W}^{\nu\sigma}\,\tilde{\mathcal{R}}_{\nu\lambda} + \mathbf{W}^{\sigma\nu}\,\tilde{\mathcal{R}}_{\lambda\nu} - 16\pi\sqrt{-g}\,\left(T^\sigma_\lambda - \frac{1}{(n-2)}T^\sigma_\sigma\delta^\sigma_\lambda\right) - 8\sqrt{-g}\,f^{\nu\sigma}A_{[\nu,\lambda]} + 2\Lambda_b\sqrt{-N}\,\delta^\sigma_\lambda + 2\Lambda_e\sqrt{-g}\,\delta^\sigma_\lambda. \tag{H.16}$$

Taking the non-covariant divergence of this also gives the Lorentz force equation,

$$\begin{aligned}
0 &= \mathbf{W}^{\sigma\nu}\,\tilde{\mathcal{R}}_{\sigma\nu,\lambda} - 16\pi\left((\sqrt{-g}\,T^\sigma_\lambda)_{,\sigma} - \frac{1}{(n-2)}(\sqrt{-g}\,T^\sigma_\sigma)_{,\lambda}\right)\\
&\quad -8\sqrt{-g}\left(-4\pi j^\nu A_{[\nu,\lambda]} + f^{\nu\sigma}A_{[\nu,\lambda],\sigma}\right) + 2\Lambda_b(\sqrt{-N})_{,\lambda} + 2\Lambda_e(\sqrt{-g})_{,\lambda} \qquad \text{(H.17)}\\
&= \mathbf{W}^{\sigma\nu}\,\tilde{\mathcal{R}}_{\sigma\nu,\lambda} - 16\pi\left((\sqrt{-g}\,T^\sigma_\lambda)_{;\sigma} - \frac{1}{2}\sqrt{-g}\,g^{\sigma\nu}{}_{,\lambda}T_{\sigma\nu} + \frac{1}{2}(\sqrt{-g}\,S^\sigma_\sigma)_{,\lambda}\right)\\
&\quad -8\sqrt{-g}\left(4\pi j^\nu A_{[\lambda,\nu]} + \frac{1}{2}f^{\nu\sigma}(3A_{[\nu,\lambda,\sigma]} - A_{[\sigma,\nu],\lambda})\right)\\
&\quad +\Lambda_b\sqrt{-N}N^{\dashv\nu\sigma}N_{\sigma\nu,\lambda} + \Lambda_e\sqrt{-g}g^{\nu\sigma}g_{\sigma\nu,\lambda} \qquad \text{(H.18)}\\
&= \mathbf{W}^{\sigma\nu}\,(\tilde{\mathcal{R}}_{\sigma\nu,\lambda} + 2A_{[\sigma,\nu],\lambda}\sqrt{2}\,i\Lambda_b^{1/2} + \Lambda_b N_{\sigma\nu,\lambda} + \Lambda_e g_{\sigma\nu,\lambda})\\
&\quad -8\pi\left(-\sqrt{-g}\,g^{\sigma\nu}{}_{,\lambda}\left(S_{\sigma\nu} - \frac{1}{2}g_{\sigma\nu}S^\alpha_\alpha\right) + (\sqrt{-g}\,S^\sigma_\sigma)_{,\lambda}\right)\\
&\quad -16\pi\sqrt{-g}\,T^\sigma_{\lambda;\sigma} - 8\sqrt{-g}\,4\pi j^\nu A_{[\lambda,\nu]} \qquad \text{(H.19)}\\
&= \mathbf{W}^{\sigma\nu}\,(\tilde{\mathcal{R}}_{\sigma\nu} + 2A_{[\sigma,\nu]}\sqrt{2}\,i\Lambda_b^{1/2} + \Lambda_b N_{\sigma\nu} + \Lambda_e g_{\sigma\nu})_{,\lambda}\\
&\quad -8\pi\left(-\sqrt{-g}\,g^{\sigma\nu}{}_{,\lambda}S_{\sigma\nu} + \frac{1}{2}\sqrt{-g}\,g^{\sigma\nu}{}_{,\lambda}g_{\sigma\nu}S^\alpha_\alpha + (\sqrt{-g}\,S^\sigma_\sigma)_{,\lambda}\right)\\
&\quad -16\pi\sqrt{-g}\left(T^\sigma_{\lambda;\sigma} + 2j^\nu A_{[\lambda,\nu]}\right) \qquad \text{(H.20)}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{W}^{\sigma\nu}\left(\tilde{\mathcal{R}}_{\sigma\nu}+2A_{[\sigma,\nu]}\sqrt{2}\,i\Lambda_b^{1/2}+\Lambda_b N_{\sigma\nu}+\Lambda_e g_{\sigma\nu}-8\pi S_{\sigma\nu}\right)_{,\lambda} \\
&\quad -8\pi\left(-\sqrt{-g}\,S^{\sigma}_{\sigma,\lambda}-(\sqrt{-g})_{,\lambda}S^{\alpha}_{\alpha}+(\sqrt{-g}\,S^{\sigma}_{\sigma})_{,\lambda}\right) \\
&\quad -16\pi\sqrt{-g}\left(T^{\sigma}_{\lambda;\sigma}+2j^{\nu}A_{[\lambda,\nu]}\right) \qquad \text{(H.21)} \\
&= -16\pi\sqrt{-g}\left(T^{\sigma}_{\lambda;\sigma}+2j^{\nu}A_{[\lambda,\nu]}\right). \qquad \text{(H.22)}
\end{aligned}
$$

It is interesting to consider the antisymmetric part of (H.11),

$$
\hat{f}^{\nu}{}_{[\sigma}\tilde{\mathcal{R}}_{[\lambda]\nu]}=4f^{\nu}{}_{[\sigma}A_{[\lambda],\nu]}. \qquad \text{(H.23)}
$$

Comparing this to the antisymmetric part of the normal field equations (2.28), we see that it implies $f^{\nu}{}_{[\sigma}N_{[\lambda]\nu]}=0$. This identity can be proven as follows,

$$
\begin{aligned}
\frac{8\sqrt{-g}}{\sqrt{-N}}\hat{f}^{\nu[\sigma}N^{[\lambda]}{}_{\nu]} &= (N^{\dashv\nu\sigma}-N^{\dashv\sigma\nu})(N^{\lambda}{}_{\nu}-N_{\nu}{}^{\lambda})-(N^{\dashv\nu\lambda}-N^{\dashv\lambda\nu})(N^{\sigma}{}_{\nu}-N_{\nu}{}^{\sigma}) \qquad \text{(H.24)} \\
&= -N^{\dashv\nu\sigma}N_{\nu}{}^{\lambda}-N^{\dashv\sigma\nu}N^{\lambda}{}_{\nu}+N^{\dashv\nu\lambda}N_{\nu}{}^{\sigma}+N^{\dashv\lambda\nu}N^{\sigma}{}_{\nu} \qquad \text{(H.25)} \\
&= \frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}[-N^{\dashv\nu\sigma}N_{\nu\rho}(N^{\dashv\rho\lambda}+N^{\dashv\lambda\rho})-N^{\dashv\sigma\nu}N_{\rho\nu}(N^{\dashv\rho\lambda}+N^{\dashv\lambda\rho}) \\
&\qquad +N^{\dashv\nu\lambda}N_{\nu\rho}(N^{\dashv\rho\sigma}+N^{\dashv\sigma\rho})+N^{\dashv\lambda\nu}N_{\rho\nu}(N^{\dashv\rho\sigma}+N^{\dashv\sigma\rho})] \qquad \text{(H.26)} \\
&= \frac{1}{2}\frac{\sqrt{-N}}{\sqrt{-g}}[-N^{\dashv\nu\sigma}N_{\nu\rho}N^{\dashv\lambda\rho}-N^{\dashv\sigma\nu}N_{\rho\nu}N^{\dashv\rho\lambda} \\
&\qquad +N^{\dashv\nu\lambda}N_{\nu\rho}N^{\dashv\sigma\rho}+N^{\dashv\lambda\nu}N_{\rho\nu}N^{\dashv\rho\sigma}]=0. \qquad \text{(H.27)}
\end{aligned}
$$

Appendix I

Alternative derivation of the $\mathcal{O}(\Lambda_b^{-1})$ field equations

Here we check the results in §2.2-§2.3 by deriving the approximate Einstein and Maxwell equations in different way, using an $\mathcal{O}(\Lambda_b^{-1})$ approximation to the Lagrangian density. Inserting the order $\mathcal{O}(\Lambda_b^{-1})$ result $\sqrt{-N} \approx \sqrt{-g}\,(1 + f^{\rho\nu}f_{\nu\rho}/\Lambda_b(n\!-\!2))$ from (C.12) into the Lagrangian density (2.10) and using (2.3) gives,

$$\begin{aligned}
\mathcal{L} \;&\approx\; -\frac{1}{16\pi}\sqrt{-N}N^{\dashv\mu\sigma}(\tilde{\mathcal{R}}_{\sigma\mu} + 2A_{[\sigma,\mu]}\sqrt{2}\,i\Lambda_b^{1/2}) \\
&\quad -\frac{1}{16\pi}(n\!-\!2)\Lambda_b\sqrt{-g}\left(1\!+\!\frac{1}{\Lambda_b(n\!-\!2)}f^{\rho\nu}f_{\nu\rho}\right) - \frac{1}{16\pi}(n\!-\!2)\Lambda_z\sqrt{-g} + \mathcal{L}_m \qquad \text{(I.1)}\\
&\approx\; -\frac{1}{16\pi}\left[\sqrt{-N}N^{\dashv\mu\sigma}(\tilde{\mathcal{R}}_{\sigma\mu} + 2A_{[\sigma,\mu]}\sqrt{2}\,i\Lambda_b^{1/2}) + (n\!-\!2)\Lambda\sqrt{-g}\right] \\
&\quad -\sqrt{-g}\frac{1}{16\pi}f^{\rho\nu}f_{\nu\rho} + \mathcal{L}_m. \qquad \text{(I.2)}
\end{aligned}$$

The $f^{\rho\nu}f_{\nu\rho}$ term looks superficially like the ordinary electromagnetic term, except that $f_{\sigma\mu}$ is defined by (2.22) rather than as $2A_{[\mu,\sigma]}$, and there is also a sign difference. The dependence of (I.2) on A_μ and $\tilde{\Gamma}^\alpha_{\sigma\mu}$ is identical to (2.10), so Ampere's law and the connection equations will be the same as usual. To take the variational derivative of (I.2) with respect to $\sqrt{-N}N^{\dashv\mu\sigma}$ it is convenient to rewrite it as,

$$\begin{aligned}\mathcal{L} \;\approx\;& -\frac{1}{16\pi}\left[\sqrt{-N}N^{\dashv\mu\sigma}(\tilde{\mathcal{R}}_{\sigma\mu}+2A_{[\sigma,\mu]}\sqrt{2}\,i\Lambda_b^{1/2})+(n-2)\Lambda\sqrt{-g}\right]\\ &-\frac{1}{16\pi}\sqrt{-g}\,\sqrt{-g}f^{\rho\nu}\sqrt{-g}f^{\alpha\tau}\frac{g_{\nu\alpha}}{\sqrt{-g}}\frac{g_{\rho\tau}}{\sqrt{-g}}+\mathcal{L}_m. \end{aligned}\tag{I.3}$$

Using

$$\frac{\partial(g_{\tau\sigma}/\sqrt{-g})}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = -\frac{g_{\tau(\nu}g_{\mu)\sigma}}{\sqrt{-g}\sqrt{-g}} \qquad \left(\text{because } \frac{\partial\left(\sqrt{-g}g^{\rho\tau}g_{\tau\sigma}/\sqrt{-g}\right)}{\partial(\sqrt{-N}N^{\dashv\mu\nu})}=0\right), \tag{I.4}$$

and (2.26,2.22) we have

$$0 = -16\pi\frac{\delta\mathcal{L}}{\delta(\sqrt{-N}N^{\dashv\mu\sigma})} \tag{I.5}$$

$$\begin{aligned} =\;& \tilde{\mathcal{R}}_{\sigma\mu}+2A_{[\sigma,\mu]}\sqrt{2}\,i\Lambda_b^{1/2}+\Lambda g_{\sigma\mu}\\ &-\sqrt{-g}\left(\delta^{[\rho}_{\mu}\delta^{\nu]}_{\sigma}\sqrt{-g}f^{\alpha\tau}+\sqrt{-g}f^{\rho\nu}\delta^{[\alpha}_{\mu}\delta^{\tau]}_{\sigma}\right)\frac{g_{\nu\alpha}}{\sqrt{-g}}\frac{g_{\rho\tau}}{\sqrt{-g}}\\ &-\sqrt{-g}\sqrt{-g}f^{\rho\nu}\sqrt{-g}f^{\alpha\tau}\left(\frac{g_{\nu(\sigma}g_{\mu)\alpha}}{\sqrt{-g}\sqrt{-g}}\frac{g_{\rho\tau}}{\sqrt{-g}}+\frac{g_{\nu\alpha}}{\sqrt{-g}}\frac{g_{\rho(\sigma}g_{\mu)\tau}}{\sqrt{-g}\sqrt{-g}}\right)\\ &+g_{\sigma\mu}\frac{1}{(n-2)}\sqrt{-g}f^{\rho\nu}\sqrt{-g}f^{\alpha\tau}\frac{g_{\nu\alpha}}{\sqrt{-g}}\frac{g_{\rho\tau}}{\sqrt{-g}}\\ &-16\pi\frac{\delta\mathcal{L}_m}{\delta(\sqrt{-N}N^{\dashv\mu\sigma})} \end{aligned}\tag{I.6}$$

$$\begin{aligned} =\;& \tilde{\mathcal{R}}_{\sigma\mu}+2A_{[\sigma,\mu]}\sqrt{2}\,i\Lambda_b^{1/2}+\Lambda g_{\sigma\mu}+\Lambda_b f_{\sigma\mu}\\ &-2\left(f_{\sigma}{}^{\nu}f_{\nu\mu}-g_{\sigma\mu}\frac{1}{2(n-2)}f^{\rho\nu}f_{\nu\rho}\right)-8\pi\left(T_{\sigma\mu}-\frac{1}{(n-2)}g_{\sigma\mu}T^{\alpha}_{\alpha}\right). \end{aligned}\tag{I.7}$$

Symmetrizing (I.7) and combining it with its contraction gives the approximate Einstein equations (2.43). Antisymmetrizing (I.7) and doing the same analysis as before gives (2.78) and Maxwell equations (2.47,2.48).

Appendix J

A weak field Lagrangian density

Here we derive an $\mathcal{O}(\Lambda_b^{-1})$ Lagrangian density for this theory which depends only on the fields $g_{\mu\nu}$, A_α and θ_α. The Lagrangian density is valid in the sense that it gives the same field equations to $\mathcal{O}(\Lambda_b^{-1})$ as calculated in §2.2-§2.4. However, the weak field Lagrangian density is derived with a somewhat ad-hoc procedure, and since it does not describe LRES theory exactly, and exact solutions would be different, one should exercise caution when using it to make definite conclusions about the theory.

Inserting the $\mathcal{O}(\Lambda_b^{-1})$ result (C.12) into our Palatini Lagrangian density and using the $\mathcal{O}(\Lambda_b^{-1})$ results (2.32,2.3,2.67,2.78,2.76) gives,

$$
\begin{aligned}
\mathcal{L}_2 &\approx -\frac{1}{16\pi}(\sqrt{-g}\, g^{\sigma\mu}+\sqrt{-g} f^{\sigma\mu}\sqrt{2}\, i\Lambda_b^{-1/2})(\tilde{\mathcal{R}}_{\sigma\mu}+2A_{[\sigma,\mu]}\sqrt{2}\, i\Lambda_b^{1/2}) \\
&\quad -\frac{1}{16\pi}(n-2)\Lambda_b\sqrt{-g}\left(1+\frac{1}{(n-2)\Lambda_b}f^{\rho\nu}f_{\nu\rho}\right)-\frac{1}{16\pi}(n-2)\Lambda_z\sqrt{-g} && \text{(J.1)} \\
&\approx -\frac{1}{16\pi}\sqrt{-g}(\tilde{\mathcal{R}}^{\mu}_{\mu}+(n-2)\Lambda-2f^{\sigma\mu}f_{\mu\sigma})-\sqrt{-g}\frac{1}{16\pi}f^{\rho\nu}f_{\nu\rho} && \text{(J.2)} \\
&\approx -\frac{\sqrt{-g}}{16\pi}\Big(R+(n-2)\Lambda-f^{\sigma\mu}f_{\mu\sigma} \\
&\qquad -\frac{2}{\Lambda_b}f^{\tau\beta}f_{\beta}{}^{\alpha}{}_{;\tau;\alpha}-\frac{n}{2(n-2)\Lambda_b}\,\ell,^{\alpha}{}_{;\alpha}-\frac{3}{2\Lambda_b}f_{[\nu\beta;\alpha]}f^{[\nu\beta}{}_{;}{}^{\alpha]} \\
&\qquad -\frac{32\pi^2 n}{(n-1)(n-2)\Lambda_b}j^{\rho}j_{\rho}-\frac{16\pi}{(n-2)\Lambda_b}f^{\alpha\tau}j_{\tau;\alpha}\Big) && \text{(J.3)} \\
&\approx -\frac{\sqrt{-g}}{16\pi}\Big(R+(n-2)\Lambda-f^{\sigma\mu}f_{\mu\sigma} \\
&\qquad -\frac{1}{\Lambda_b}f^{\beta\tau}\left(\acute{C}_{\beta\tau\alpha\rho}f^{\alpha\rho}+8\pi j_{[\beta,\tau]}\right)-\frac{n}{2(n-2)\Lambda_b}\,\ell,^{\alpha}{}_{;\alpha}+\frac{4}{\Lambda_b}\theta^{\rho}\theta_{\rho} \\
&\qquad -\frac{32\pi^2 n}{(n-1)(n-2)\Lambda_b}j^{\rho}j_{\rho}-\frac{16\pi}{(n-2)\Lambda_b}f^{\alpha\tau}j_{\tau;\alpha}\Big) && \text{(J.4)} \\
&\approx -\frac{\sqrt{-g}}{16\pi}\Big(R+(n-2)\Lambda-f^{\sigma\mu}\not{F}_{\mu\sigma}-\frac{n}{2(n-2)\Lambda_b}\,\ell,^{\alpha}{}_{;\alpha}+\frac{4}{\Lambda_b}\theta^{\rho}\theta_{\rho} \\
&\qquad -\frac{32\pi^2 n}{(n-1)(n-2)\Lambda_b}j^{\rho}j_{\rho}+\frac{8\pi(n-4)}{(n-2)\Lambda_b}f^{\alpha\tau}j_{\tau;\alpha}\Big) && \text{(J.5)} \\
&\approx -\frac{\sqrt{-g}}{16\pi}\Big(R+(n-2)\Lambda-f^{\sigma\mu}\not{F}_{\mu\sigma}-\frac{n}{2(n-2)\Lambda_b}\,\ell,^{\alpha}{}_{;\alpha}+\frac{4}{\Lambda_b}\theta^{\rho}\theta_{\rho} \\
&\qquad -\frac{32\pi^2 n}{(n-1)(n-2)\Lambda_b}j^{\rho}j_{\rho}+\frac{8\pi(n-4)}{(n-2)\Lambda_b}\left((f^{\alpha\tau}j_{\tau})_{;\alpha}-4\pi j^{\tau}j_{\tau}\right)\Big) && \text{(J.6)} \\
&\approx -\frac{\sqrt{-g}}{16\pi}\Big(R+(n-2)\Lambda-f^{\sigma\mu}\not{F}_{\mu\sigma}-\frac{32\pi^2(n-2)}{(n-1)\Lambda_b}j^{\rho}j_{\rho}+\frac{4}{\Lambda_b}\theta^{\rho}\theta_{\rho} \\
&\qquad -\frac{n}{2(n-2)\Lambda_b}\,\ell,^{\alpha}{}_{;\alpha}+\frac{8\pi(n-4)}{(n-2)\Lambda_b}(f^{\alpha\tau}j_{\tau})_{;\alpha}\Big). && \text{(J.7)}
\end{aligned}
$$

where we write (2.78) as

$$
\begin{aligned}
f_{\sigma\mu} &= \not{F}_{\sigma\mu}+\frac{1}{\Lambda_b}f^{\rho\tau}\acute{C}_{\rho\tau\sigma\mu}, && \text{(J.8)} \\
\not{F}_{\sigma\mu} &= 2A_{[\mu,\sigma]}+\frac{1}{\Lambda_b}\theta_{[\tau,\alpha]}\varepsilon_{\sigma\mu}{}^{\tau\alpha}+\frac{8\pi(n-2)}{\Lambda_b(n-1)}j_{[\sigma,\mu]}, && \text{(J.9)} \\
\acute{C}_{\sigma\mu\alpha\rho} &= R_{\sigma\mu\alpha\rho}-g_{\sigma[\alpha}R_{\rho]\mu}+g_{\mu[\alpha}R_{\rho]\sigma}. && \text{(J.10)}
\end{aligned}
$$

Removing the total divergences gives

$$\mathcal{L}_2 \;\approx\; -\frac{\sqrt{-g}}{16\pi}\left(R+(n-2)\Lambda-f^{\sigma\mu}\!F_{\mu\sigma}+\frac{4}{\Lambda_b}\theta^\rho\theta_\rho\right)+\sqrt{-g}\frac{2\pi(n-2)}{(n-1)\Lambda_b}j^\rho j_\rho. \quad \text{(J.11)}$$

Now let us redefine the electromagnetic potential

$$\breve{A}_\mu = A_\mu - \frac{4\pi(n-2)}{\Lambda_b(n-1)}j_\mu. \quad \text{(J.12)}$$

In terms of this shifted potential, $f_{\mu\nu}$ and $F_{\mu\nu}$ from (2.78) lose their $j_{[\sigma,\mu]}$ terms and Maxwell's equations (2.47,2.48) become more exact. This redefinition brings the same $j^\sigma j_\sigma$ term out of $\mathcal{L}_m$ for all of the $\mathcal{L}_m$ cases. For the classical hydrodynamics case (L.1),

$$\begin{aligned}
\mathcal{L}_m &= -\frac{\boldsymbol{\mu}Q}{m}u^\sigma A_\sigma - \frac{\boldsymbol{\mu}}{2}u^\alpha g_{\alpha\nu}u^\nu, && \text{(J.13)}\\
j^\alpha &= \frac{\boldsymbol{\mu}Q}{m\sqrt{-g}}u^\alpha, && \text{(J.14)}\\
\Delta\mathcal{L}_m &= -\sqrt{-g}\frac{4\pi(n-2)}{\Lambda_b(n-1)}j^\sigma j_\sigma. && \text{(J.15)}
\end{aligned}$$

For the spin-0 case (L.2),

$$\begin{aligned}
\mathcal{L}_m &= \sqrt{-g}\,\frac{1}{2}\left(\frac{\hbar^2}{m}\bar\psi\overleftarrow{D}_\mu D^\mu\psi - m\bar\psi\psi\right), && \text{(J.16)}\\
D_\mu &= \frac{\partial}{\partial x^\mu}+\frac{iQ}{\hbar}A_\mu \;\;,\;\; \overleftarrow{D}_\mu = \frac{\overleftarrow{\partial}}{\partial x^\mu}-\frac{iQ}{\hbar}A_\mu, && \text{(J.17)}\\
j^\alpha &= \frac{i\hbar Q}{2m}(\bar\psi D^\alpha\psi - \bar\psi\overleftarrow{D}^\alpha\psi), && \text{(J.18)}\\
\Delta\mathcal{L}_m &= \sqrt{-g}\frac{1}{2}\frac{\hbar^2}{m}\left(-\frac{iQ}{\hbar}\right)\frac{2m}{i\hbar Q}\frac{4\pi(n-2)}{\Lambda_b(n-1)}j^\sigma j_\sigma = -\sqrt{-g}\frac{4\pi(n-2)}{\Lambda_b(n-1)}j^\sigma j_\sigma. && \text{(J.19)}
\end{aligned}$$

For the spin-1/2 case (L.4),

$$\mathcal{L}_m = \sqrt{-g}\left(\frac{i\hbar}{2}(\bar\psi\gamma^\nu D_\nu\psi - \bar\psi\overleftarrow{D}_\nu\gamma^\nu\psi) - mc^2\bar\psi\psi\right), \tag{J.20}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} + \tilde\Gamma_\mu + \frac{iQ}{\hbar}A_\mu \;\; , \;\; \overleftarrow{D}_\mu = \frac{\overleftarrow{\partial}}{\partial x^\mu} + \tilde\Gamma^\dagger_\mu - \frac{iQ}{\hbar}A_\mu, \tag{J.21}$$

$$j^\alpha = Q\bar\psi\gamma^\alpha\psi, \tag{J.22}$$

$$\Delta\mathcal{L}_m = \sqrt{-g}\, i\hbar\bar\psi\gamma^\sigma\psi\left(\frac{iQ}{\hbar}\right)\frac{4\pi(n-2)}{\Lambda_b(n-1)}j_\sigma = -\sqrt{-g}\frac{4\pi(n-2)}{\Lambda_b(n-1)}j^\sigma j_\sigma. \tag{J.23}$$

Here $l_P = \sqrt{\hbar G/c^3}$ is the Planck length. The $\Delta\mathcal{L}_m$ contribution is halved by the $j^\sigma j_\sigma$ term in the original Lagrangian density so that the total shifted Lagrangian density is

$$\breve{\mathcal{L}}_2 \approx -\frac{\sqrt{-g}}{16\pi}\left(R + (n-2)\Lambda - \breve{f}^{\sigma\mu}\breve{F}_{\mu\sigma} + \frac{4}{\Lambda_b}\theta^\rho\theta_\rho\right) - \sqrt{-g}\frac{2\pi(n-2)}{\Lambda_b(n-1)}j^\rho j_\rho$$
$$+\breve{\mathcal{L}}_{hydrodynamics} + \breve{\mathcal{L}}_{spin-0} + \breve{\mathcal{L}}_{spin-1/2} \;\ldots. \tag{J.24}$$

where $\breve{f}^{\sigma\mu}$ and $\breve{F}_{\mu\sigma}$ are (J.8,J.9) but without the j_α terms,

$$\breve{f}_{\sigma\mu} = \breve{F}_{\sigma\mu} + \frac{1}{\Lambda_b}\breve{f}^{\alpha\rho}\acute{C}_{\sigma\mu\alpha\rho}, \tag{J.25}$$

$$\breve{F}_{\sigma\mu} = 2\breve{A}_{[\mu,\sigma]} + \frac{1}{\Lambda_b}\theta_{[\tau,\alpha]}\varepsilon_{\sigma\mu}{}^{\tau\alpha}. \tag{J.26}$$

Expanding things out and ignoring higher order powers of $C_{\sigma\mu\alpha\rho}/\Lambda_b$ and total divergences, our effective weak-field Lagrangian density becomes

$$\begin{aligned}
\breve{\mathcal{L}}_2 \approx\; & -\frac{\sqrt{-g}}{16\pi}(R + (n-2)\Lambda) && \begin{pmatrix}\text{gravitational}\\ \text{terms}\end{pmatrix}\\
& +\frac{\sqrt{-g}}{4\pi}\breve{A}^{[\mu}{}_{;}{}^{\sigma]}\breve{A}_{[\sigma,\mu]} - \sqrt{-g}\frac{2\pi(n-2)}{\Lambda_b(n-1)}j^\rho j_\rho && \begin{pmatrix}\text{electromagnetic}\\ \text{terms}\end{pmatrix}\\
& -\frac{\sqrt{-g}}{4\pi}\frac{1}{\Lambda_b^2}\left(\theta^{[\mu}{}_{;}{}^{\sigma]}\theta_{[\sigma,\mu]} + \Lambda_b\theta^\rho\theta_\rho\right) && \begin{pmatrix}\text{Proca}\\ \text{terms}\end{pmatrix}\\
& -\frac{\sqrt{-g}}{4\pi}\left(\breve{A}_{[\mu,\sigma]} + \frac{1}{2\Lambda_b}\theta_{[\tau,\alpha]}\varepsilon_{\sigma\mu}{}^{\tau\alpha}\right)\frac{C^{\mu\sigma\rho\nu}}{\Lambda_b}\left(\breve{A}_{[\rho,\nu]} + \frac{1}{2\Lambda_b}\theta_{[\beta,\lambda]}\varepsilon_{\rho\nu}{}^{\beta\lambda}\right) && \begin{pmatrix}\text{coupling}\\ \text{terms}\end{pmatrix}\\
& +\breve{\mathcal{L}}_{hydrodynamics} + \breve{\mathcal{L}}_{spin-0} + \breve{\mathcal{L}}_{spin-1/2} \;\ldots.
\end{aligned} \tag{J.27}$$

Even though we call this a weak-field Lagrangian density, it only neglects terms which are $< 10^{-64}$ of the leading order terms for worst-case field strengths. We can probably assume $\theta_\alpha \approx 0$ because Proca plane waves would have a minimum frequency $\omega_{Proca} = \sqrt{2\Lambda_b} \sim \sqrt{2}\omega_c^2 l_P \sim \sqrt{2}/l_P$ which exceeds the zero-point cut-off frequency $\omega_c \sim 1/l_P$. Alternatively this field could function as a built-in Pauli-Villars field as discussed in Appendix K.

Finally, note that if we take the variational derivative of our shifted Lagrangian density (J.27) with respect to $\bar{\psi}$ we get the unshifted Dirac equation,

$$
\begin{aligned}
0 &= \frac{1}{c\sqrt{-g}}\left[\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \left(\frac{\partial\mathcal{L}}{\partial\bar{\psi}_{,\lambda}}\right)_{,\lambda}\right] &\text{(J.28)}\\
&= \frac{2\pi c^2(n-2)Q}{G(n-1)}\psi\gamma^\rho j_\rho + i\hbar\gamma^\nu \breve{D}_\nu\psi - mc\psi &\text{(J.29)}\\
&= i\hbar\gamma^\nu D_\nu\psi - mc\psi. &\text{(J.30)}
\end{aligned}
$$

Presumably the same thing occurs with the Klein-Gordon equation. This should be expected because our shift in the electromagnetic potential is only a redefinition and should not result in different field equations.

Now let us derive the field equations from the weak field Lagrangian density for the source-free case. The $\mathcal{O}(\Lambda_b^{-1})$ Maxwell equations (2.47,2.48) can be derived by setting $\delta\mathcal{L}_2/\delta A_\nu = 0$. Using (J.10,J.9,J.11) and

$$
\begin{aligned}
f^{\sigma\mu} &= \not{I}^{\dashv\sigma\mu\alpha\rho}\not{F}_{\alpha\rho}, &\text{(J.31)}\\
\not{I}_{\alpha\rho\sigma\mu} &= g_{\alpha[\sigma}g_{\mu]\rho} - \acute{C}_{\alpha\rho\sigma\mu}/\Lambda_b &\text{(J.32)}\\
\delta(f^{\sigma\mu}\not{F}_{\mu\sigma}) &= 2f^{\sigma\mu}\delta\not{F}_{\mu\sigma} + f^{\sigma\mu}f^{\alpha\rho}\delta\not{I}_{\sigma\mu\alpha\rho}, &\text{(J.33)}
\end{aligned}
$$

gives

$$0 = -8\pi\frac{\delta\mathcal{L}_2}{\delta A_\nu} = -\sqrt{-g}f^{\sigma\mu}\frac{\delta F_{\mu\sigma}}{\delta A_\nu} = \left(2\sqrt{-g}f^{\sigma\mu}\frac{\partial A_{[\sigma\mu]}}{\partial A_{\nu,\omega}}\right)_{,\omega} = 2(\sqrt{-g}f^{\nu\omega})_{,\omega}. \quad (J.34)$$

From this and (2.21) we get Maxwell's equations (2.47,2.48) as before.

The $\mathcal{O}(\Lambda_b^{-1})$ Proca equation (2.81) can be derived by setting $\delta\mathcal{L}_2/\delta\theta_\nu = 0$. Using (J.33,J.32,J.10,J.9,J.8,J.11) gives

$$\begin{aligned}
0 &= -8\pi\Lambda_b^2\frac{\delta\mathcal{L}_2}{\delta\theta_\nu} && (J.35)\\
&= -\Lambda_b^2\sqrt{-g}f^{\sigma\mu}\frac{\delta F_{\mu\sigma}}{\delta\theta_\nu} + \sqrt{-g}\,4\Lambda_b\theta^\nu && (J.36)\\
&= \Lambda_b\left(\sqrt{-g}f^{\sigma\mu}\frac{\partial(\varepsilon_{\mu\sigma}{}^{\rho\tau}\theta_{\rho,\tau})}{\partial\theta_{\nu,\omega}}\right)_{,\omega} + \sqrt{-g}\,4\Lambda_b\theta^\nu && (J.37)\\
&= \Lambda_b(\sqrt{-g}f^{\sigma\mu}\varepsilon_{\mu\sigma}{}^{\nu\omega})_{,\omega} + \sqrt{-g}\,4\Lambda_b\theta^\nu && (J.38)\\
&= 2\Lambda_b\sqrt{-g}(A_{\mu,\sigma}\varepsilon^{\mu\sigma\nu\omega})_{;\omega} + (\sqrt{-g}\,(4\theta^{[\nu}{}_;{}^{\omega]} + f^{\rho\tau}\acute{C}_{\rho\tau\sigma\mu}\varepsilon^{\mu\sigma\nu\omega}))_{,\omega} + \sqrt{-g}\,4\Lambda_b\theta^\nu && (J.39)\\
&= 4(\sqrt{-g}\theta^{[\nu}{}_;{}^{\omega]})_{,\omega} - \sqrt{-g}\,\varepsilon^{\nu\omega\sigma\mu}(f^{\rho\tau}\acute{C}_{\rho\tau[\sigma\mu)},_{\omega]} + \sqrt{-g}\,4\Lambda_b\theta^\nu. && (J.40)
\end{aligned}$$

From this we get the Proca equation (2.81) as before.

The $\mathcal{O}(\Lambda_b^{-1})$ Einstein equations (2.42) can be derived by setting $\delta\mathcal{L}_2/\delta g_{\sigma\mu} = 0$. First we will deal with the $\acute{C}_{\lambda\rho\alpha\nu}$ term in (J.33,J.32). From (J.10,2.76) we have

$$\begin{aligned}
f^{\lambda\rho}f^{\alpha\nu}\delta\acute{C}_{\lambda\rho\alpha\nu} &= f^{\lambda\rho}f^{\alpha\nu}\delta(R_{\lambda\rho\alpha\nu} - g_{\lambda[\alpha}R_{\nu]\rho} + g_{\rho[\alpha}R_{\nu]\lambda}) && (J.41)\\
&= f_\tau{}^\rho f^{\alpha\nu}\delta(R^\tau{}_{\rho\alpha\nu} - 2\delta^\tau_\alpha R_{\nu\rho}) + f^{\lambda\rho}f^{\alpha\nu}\delta g_{\lambda\tau}(R^\tau{}_{\rho\alpha\nu} - 2\delta^\tau_\alpha R_{\nu\rho}) && (J.42)\\
&= f_\tau{}^\rho f^{\alpha\nu}\delta(R^\tau{}_{\rho\alpha\nu} - 2\delta^\tau_\alpha R_{\nu\rho}) + 2\delta g_{\lambda\tau}f^{\lambda\rho}f^{\alpha\tau}{}_{;\rho;\alpha}. && (J.43)
\end{aligned}$$

To calculate $f_\tau{}^\rho f^{\alpha\nu}\delta(R^\tau{}_{\rho\alpha\nu}–2\delta^\tau_\alpha R_{\nu\rho})$ we will assume locally geodesic coordinates where $\Gamma^\rho_{\sigma\mu} = 0$. With this method, terms with a $\Gamma^\rho_{\sigma\mu}$ factor can be ignored, and covariant derivatives are equivalent to ordinary derivatives, as long as they are not inside a derivative. Then from the definition of the Ricci tensor we have

$$f_\tau{}^\rho f^{\alpha\nu}\delta(R^\tau{}_{\rho\alpha\nu}-2\delta^\tau_\alpha R_{\nu\rho}) \;=\; 2f_\tau{}^\rho f^{\alpha\nu}\delta\Gamma^\tau_{\rho[\nu,\alpha]}+4\,\ell^{\rho\nu}\delta\Gamma^\alpha_{\nu[\rho,\alpha]} \tag{J.44}$$

$$= 2f_\tau{}^\rho f^{\alpha\nu}\delta\Gamma^\tau_{\rho\nu,\alpha}+2\,\ell^{\rho\nu}\delta\Gamma^\alpha_{\nu\rho,\alpha}-2\,\ell^{\rho\nu}\delta\Gamma^\alpha_{\nu\alpha,\rho} \tag{J.45}$$

$$= 2f_\tau{}^\rho f^{\alpha\nu}(\delta\Gamma^\tau_{\rho\nu})_{;\alpha}+2\,\ell^{\rho\nu}(\delta\Gamma^\alpha_{\nu\rho})_{;\alpha}-2\,\ell^{\rho\nu}(\delta\Gamma^\alpha_{\nu\alpha})_{;\rho} \tag{J.46}$$

$$= 2(f_\tau{}^\rho f^{\alpha\nu}\delta\Gamma^\tau_{\rho\nu})_{;\alpha}+2(\,\ell^{\rho\nu}\delta\Gamma^\alpha_{\nu\rho})_{;\alpha}-2(\,\ell^{\rho\nu}\delta\Gamma^\alpha_{\nu\alpha})_{;\rho}$$

$$-2(f_\tau{}^\rho f^{\alpha\nu})_{;\alpha}\delta\Gamma^\tau_{\rho\nu}-2\,\ell^{\rho\nu}{}_{;\alpha}\delta\Gamma^\alpha_{\nu\rho}+2\,\ell^{\rho\nu}{}_{;\rho}\delta\Gamma^\alpha_{\nu\alpha}, \tag{J.47}$$

where

$$\ell^{\rho\nu}=f^\rho{}_\tau f^{\tau\nu}. \tag{J.48}$$

The first line of (J.47) is the divergence of a vector, so assuming that $\delta\Gamma^\alpha_{\nu\rho}=0$ on the boundary of integration, these terms can be dropped. Substituting the Christoffel connection (2.20) into the remaining terms gives

$$f_\tau{}^\rho f^{\alpha\nu}\delta(R^\tau{}_{\rho\alpha\nu}-2\delta^\tau_\alpha R_{\nu\rho}) = -(f_\tau{}^\rho f^{\alpha\nu})_{;\alpha}g^{\tau\beta}(\delta g_{\nu\beta,\rho}+\delta g_{\beta\rho,\nu}-\delta g_{\rho\nu,\beta})$$

$$-\;\ell^{\rho\nu}{}_{;\alpha}g^{\alpha\beta}(\delta g_{\rho\beta,\nu}+\delta g_{\beta\nu,\rho}-\delta g_{\nu\rho,\beta})$$

$$+\;\ell^{\rho\nu}{}_{;\rho}g^{\alpha\beta}(\delta g_{\alpha\beta,\nu}+\delta g_{\beta\nu,\alpha}-\delta g_{\nu\alpha,\beta}) \tag{J.49}$$

$$= -(f^{\beta\rho}f^{\alpha\nu})_{;\alpha}2\delta g_{\nu\beta,\rho}-\;\ell^{\rho\nu}{}_;{}^\beta(2\delta g_{\rho\beta,\nu}-\delta g_{\nu\rho,\beta})+\;\ell^{\rho\nu}{}_{;\rho}g^{\alpha\beta}\delta g_{\alpha\beta,\nu} \tag{J.50}$$

$$= -(f^{\beta\rho}f^{\alpha\nu})_{;\alpha}2(\delta g_{\nu\beta})_{;\rho}-\;\ell^{\rho\nu}{}_;{}^\beta(2(\delta g_{\rho\beta})_{;\nu}-(\delta g_{\nu\rho})_{;\beta})+\;\ell^{\rho\nu}{}_{;\rho}g^{\alpha\beta}(\delta g_{\alpha\beta})_{;\nu} \tag{J.51}$$

$$= -2((f^{\beta\rho}f^{\alpha\nu})_{;\alpha}\delta g_{\nu\beta})_{;\rho}-2(\,\ell^{\rho\nu}{}_;{}^\beta\delta g_{\rho\beta})_{;\nu}+(\,\ell^{\rho\nu}{}_;{}^\beta\delta g_{\nu\rho})_{;\beta}+(\,\ell^{\rho\nu}{}_{;\rho}g^{\alpha\beta}\delta g_{\alpha\beta})_{;\nu}$$

$$+\,2(f^{\beta\rho}f^{\alpha\nu})_{;\alpha;\rho}\delta g_{\nu\beta}+2\,\ell^{\rho\nu}{}_;{}^\beta{}_{;\nu}\delta g_{\rho\beta}-\;\ell^{\rho\nu}{}_;{}^\beta{}_{;\beta}\delta g_{\nu\rho}-\;\ell^{\rho\nu}{}_{;\rho;\nu}g^{\alpha\beta}\delta g_{\alpha\beta}. \tag{J.52}$$

The first line of (J.52) is the divergence of a vector, so assuming that $\delta g_{\sigma\mu}=0$ on the boundary of integration, these terms can be dropped. Using (J.52,J.43,J.33,J.32,J.9, 2.79,2.76,J.11) and assuming a covariant θ_ρ, the terms of $\delta\mathcal{L}_2/\delta g_{\sigma\mu}$ are then

$$\frac{-1}{2\Lambda_b} f^{\lambda\rho} f^{\alpha\nu} \frac{\delta \acute{C}_{\lambda\rho\alpha\nu}}{\delta g_{\sigma\mu}} = \frac{-1}{2\Lambda_b} \Big((f^{\mu\rho} f^{\alpha\sigma})_{;\alpha;\rho} + (f^{\sigma\rho} f^{\alpha\mu})_{;\alpha;\rho} + \ell^{\sigma\nu}{}_{;}{}^{\mu}{}_{;\nu} + \ell^{\mu\nu}{}_{;}{}^{\sigma}{}_{;\nu} - \ell^{\mu\sigma}{}_{;}{}^{\beta}{}_{;\beta} - \ell^{\rho\nu}{}_{;\rho;\nu} g^{\sigma\mu} + f^{\sigma\rho} f^{\alpha\mu}{}_{;\rho;\alpha} + f^{\mu\rho} f^{\alpha\sigma}{}_{;\rho;\alpha} \Big), \quad \text{(J.53)}$$

$$\frac{1}{2}\left(2 f^{\alpha\rho} \frac{\delta \breve{F}_{\rho\alpha}}{\delta g_{\sigma\mu}} \right) = \frac{1}{\Lambda_b} \left(f^{\sigma\rho}\, \varepsilon_{\rho}{}^{\mu\tau\nu} \theta_{\tau,\nu} + f^{\mu\rho}\, \varepsilon_{\rho}{}^{\sigma\tau\nu} \theta_{\tau,\nu} - \frac{1}{2} g^{\sigma\mu} f^{\alpha\rho}\, \varepsilon_{\rho\alpha}{}^{\tau\nu} \theta_{\tau,\nu} \right) \quad \text{(J.54)}$$

$$= \frac{1}{\Lambda_b} \left(-\frac{3}{2} f^{\sigma}{}_{\rho}\, f^{[\rho\mu}{}_{;}{}^{\nu]}{}_{;\nu} - \frac{3}{2} f^{\mu}{}_{\rho}\, f^{[\rho\sigma}{}_{;}{}^{\nu]}{}_{;\nu} + \frac{3}{4} g^{\sigma\mu} f_{\alpha\rho} f^{[\rho\alpha}{}_{;}{}^{\nu]}{}_{;\nu} \right) \quad \text{(J.55)}$$

$$-\frac{2}{\Lambda_b} \frac{\delta(\theta^\rho\theta_\rho)}{\delta g_{\sigma\mu}} = \frac{2}{\Lambda_b}\, \theta^\sigma\theta^\mu = \frac{1}{\Lambda_b} \left(\frac{9}{4}\, g^{\sigma\rho} f_{[\nu\beta;\rho]} f^{[\nu\beta}{}_{;}{}^{\mu]} + 2 g^{\sigma\mu}\theta^\rho\theta_\rho \right) \quad \text{(J.56)}$$

$$= \frac{1}{\Lambda_b} \left(\frac{1}{4} (f_{\nu\beta;}{}^{\sigma} + 2 f^{\sigma}{}_{\nu;\beta}) (f^{\nu\beta}{}_{;}{}^{\mu} + f^{\mu\nu}{}_{;}{}^{\beta} + f^{\beta\mu}{}_{;}{}^{\nu}) + 2 g^{\sigma\mu}\theta^\rho\theta_\rho \right) \quad \text{(J.57)}$$

$$= \frac{1}{\Lambda_b} \left(\frac{1}{4} f_{\nu\beta;}{}^{\sigma} f^{\nu\beta}{}_{;}{}^{\mu} + \frac{1}{2} f_{\nu\beta;}{}^{\sigma} f^{\beta\mu}{}_{;}{}^{\nu} + \frac{1}{2} f^{\sigma}{}_{\nu;\beta} f^{\nu\beta}{}_{;}{}^{\mu} + \frac{1}{2} f^{\sigma}{}_{\nu;\beta} f^{\mu\nu}{}_{;}{}^{\beta} + \frac{1}{2} f^{\sigma}{}_{\nu;\beta} f^{\beta\mu}{}_{;}{}^{\nu} - \frac{1}{4} g^{\sigma\mu} f_{\nu\beta;\alpha} f^{\nu\beta}{}_{;}{}^{\alpha} - \frac{1}{2} g^{\sigma\mu} f_{\nu\beta;\alpha} f^{\alpha\nu}{}_{;}{}^{\beta} \right), \quad \text{(J.58)}$$

$$-\frac{2}{\Lambda_b} \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g_{\sigma\mu}} \theta^\rho\theta_\rho = -\frac{1}{\Lambda_b} g^{\sigma\mu}\theta^\rho\theta_\rho = \frac{3}{8\Lambda_b} g^{\sigma\mu} f_{[\nu\beta;\alpha]} f^{[\nu\beta}{}_{;}{}^{\alpha]}, \quad \text{(J.59)}$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g_{\sigma\mu}}\, f^{\alpha\rho} \breve{F}_{\rho\alpha} \right) = \frac{1}{4} g^{\sigma\mu} f^{\alpha\rho} \breve{F}_{\rho\alpha} \quad \text{(J.60)}$$

$$= \frac{1}{4} g^{\sigma\mu} f^{\alpha\rho} \left(f_{\rho\alpha} - \frac{1}{\Lambda_b} f^{\tau\nu} \acute{C}_{\tau\nu\rho\alpha} \right) \quad \text{(J.61)}$$

$$= \frac{1}{4} g^{\sigma\mu} f^{\alpha\rho} f_{\rho\alpha} - \frac{1}{2\Lambda_b} g^{\sigma\mu} f^{\alpha\rho} f^{\beta}{}_{[\rho;\alpha];\beta}, \quad \text{(J.62)}$$

$$\frac{1}{2} f^{\lambda\rho} f^{\alpha\nu} \frac{\delta(g_{\lambda[\alpha} g_{\nu]\rho})}{\delta g_{\sigma\mu}} = f^{\sigma\rho} f^{\mu}{}_{\rho}, \quad \text{(J.63)}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}((n-2)\Lambda + R))}{\delta g_{\sigma\mu}} = \frac{(n-2)}{2} \Lambda g^{\sigma\mu} - G^{\sigma\mu}. \quad \text{(J.64)}$$

Setting the sum of these terms to zero and applying Ampere's law (2.47) yields the order $\mathcal{O}(\Lambda_b^{-1})$ Einstein equations (2.42).

Appendix K

Proca-waves as Pauli-Villars ghosts?

Here we investigate the possibility that the θ_ν field in (2.81,J.27) could function as a built-in Pauli-Villars field. Recall that in quantum electrodynamics, a cutoff wavenumber is often not implemented by simply substituting k_c for ∞ in the upper limit of integrals. Instead, Pauli-Villars regularization is often used because it is Lorentz invariant. With the Pauli-Villars method, a ficticious particle is introduced into the Lagrangian which has a huge mass, M say, and which has the opposite sign in the Lagrangian, meaning that it is a ghost, with negative norm or negative energy. This has the same effect as a cutoff wavenumber where $k_c = M$ (in quantum electrodynamics natural units). To calculate the electron self-energy for example, this ficticious particle is a ghost Proca particle. When calculating the amplitude of a process involving a photon, the integral associated with every Feynman graph will contain a Feynman propagator of the form

$$D_{ij} = \frac{g_{ij}}{(p-k)^2 + i\epsilon}. \tag{K.1}$$

Introducing the ghost Proca particle means that for every Feynman graph that one had originally, there will be a new one where the associated integral contains instead

$$D_{ij} = -\frac{g_{ij}}{(p-k)^2 - M^2 + i\epsilon} \qquad (\textit{note the minus sign}). \tag{K.2}$$

Because of the way that amplitudes are combined, this has the effect of replacing the original photon propagator in all Feynman graphs with the sum of the two above

$$D_{ij} = \frac{g_{ij}}{(p-k)^2 + i\epsilon} - \frac{g_{ij}}{(p-k)^2 - M^2 + i\epsilon}. \tag{K.3}$$

When one integrates over k, the two parts cancel each other for $k >> M$, effectively cutting off the integral. However, because the additional term of the propagator has the huge mass M in its denominator, this term has virtually no effect for ordinary momenta that can be produced in accelerators and astrophysical phenomena.

The point of this is that the ghost Proca particle that seems to come out of the theory is just what is needed for Pauli-Villars regularization. This can be seen from the effective weak-field Lagrangian density for the theory in Appendix J. All that would necessary to do would be to include a coupling of this particle with the electron when spin-1/2 particles are added to the theory. So this ghost Proca particle could actually be a blessing in disguise, because it potentially frees the theory from divergences which must be removed artificially in ordinary quantum electrodynamics. This also illustrates that a ghost particle with mass near the inverse Planck length is a whole different animal than one with an ordinary mass.

With further work, it might even be possible to free the theory of its reliance on an externally imposed cutoff frequency. The Pauli-Villars cutoff caused by the ghost Proca particle should also cutoff the calculation of Λ_b (for photons anyway). Then combing (2.12,2.81) we get

$$\omega_c = k_c = \sqrt{2\Lambda_b}, \tag{K.4}$$

$$\Lambda_b = C_z l_P^2 \omega_c^4 = C_z l_P^2 (2\Lambda_b)^2, \tag{K.5}$$

$$\Rightarrow \Lambda_b = \frac{1}{4C_z l_P^2}, \tag{K.6}$$

where l_P is the Planck length, ω_c is the cutoff frequency (2.13), and C_z comes from (2.14). But we have calculated in (7.15) what Λ_b must be in the non-Abelian theory, and presumably this should apply for the Abelian theory also. Equating the values from (7.15) and (K.6) gives

$$\Lambda_b = \frac{1}{4C_z l_P^2} = \frac{\alpha}{8 l_P^2 sin^2\theta_w} \quad \Rightarrow \quad C_z = \frac{2sin^2\theta_w}{\alpha}. \tag{K.7}$$

Using $\alpha = e^2/\hbar c = 1/137.036$, $\sin^2\theta_w = .2397 \pm .0013$ and the definition (2.14) gives

$$\left(\begin{matrix}\text{fermion}\\ \text{spin states}\end{matrix} - \begin{matrix}\text{boson}\\ \text{spin states}\end{matrix}\right) = \frac{4\pi sin^2\theta_w}{\alpha} = 412.8 \pm 2. \tag{K.8}$$

The result (K.8) is interesting, partly because the theory predicts that the difference should be an integer, and this potentially allows the theory to be proven or disproven. At present the weak mixing angle θ_w cannot be measured accurately enough to determine whether we are seeing an integer or not. The issue is also complicated because the value of $sin^2\theta_w/\alpha$ "runs", meaning that its value depends logarithmically on the energy at which it is measured. To really do an accurate calculation we would need to use its value at the same cutoff frequency ω_c used to calculate Λ_z, but this could be done. It is likely that measurement accuracy will improve enough in the near future so that we can determine whether (K.8) is consistent with an integer. The result (K.8) is also interesting because it might select among the different possibilities of matrix size for the non-Abelian version of our theory. For the non-Abelian theory we used 2×2 matrices in order to get Einstein-Weinberg-Salam theory, but we had the choice of using any matrix size "d", corresponding to $U(1)\otimes SU(d)$ instead of $U(1)\otimes SU(2)$. Each choice of "d" will result in different numbers of fermion and boson spin states. It would be very nice if some choice of "d" agreed with (K.8). The choice $d=5$ with $U(1)\otimes SU(5)$ is particularly interesting because $SU(5)$ has long been considered as a way of unifying the strong and weak forces in the $U(1)\otimes SU(2)\otimes SU(3)$ gauge structure of the Standard Model. However, the calculation of the left-hand side of (K.8) is complicated, and it is unclear whether to include scalar particles and gravitons, and it is even more unclear how to account for possible additional particles associated with a non-Abelian $g_{\nu\mu}$. For the Standard Model the left-hand side of (K.8) works out to about 60.

To do this rigorously, we would also need to include Pauli-Villars ghosts corresponding to electrons. Of course the theory only approximates electro-vac Einstein-Maxwell theory, so we must add in spin-1/2 particles (one can think of this as 1st quantization of an electric monopole solution). In any case, for every spin-1/2 particle that are added to the theory, it would be easy to also add in a corresponding Pauli-Villars ghost. Surely having Pauli-Villars ghosts as an inherent part of quantum electrodynamics can't be any worse than introducing ficticious particles just to make divergent integrals come out finite, and then forgetting about them.

Appendix L

$\mathcal{L}_m$, $T_{\mu\nu}$, j^μ and kinetic equations for spin-0 and spin-1/2 sources

Here we display the matter Lagrangian $\mathcal{L}_m$ for the classical hydrodynamics, spin-0 and spin-1/2 cases, and we derive the energy-momentum tensors and charge currents for each. Then we derive the Klein-Gordon equation and Dirac equation, and we derive the continuity equation and Lorentz force equation from the Klein-Gordon equation. All of these results are shown to be identical to ordinary Einstein-Maxwell theory and one-particle quantum mechanics.

For the classical hydrodynamics case we can form a rather artificial $\mathcal{L}_m$ which depends on a mass scalar density $\boldsymbol{\mu}$ and a velocity vector u^ν, neither of which is constrained (that is we will not require $\delta\mathcal{L}/\delta\boldsymbol{\mu} = 0$ or $\delta\mathcal{L}/\delta u^\sigma = 0$),

$$\mathcal{L}_m \quad = \quad -\frac{\boldsymbol{\mu} Q}{m} u^\nu A_\nu - \frac{\boldsymbol{\mu}}{2} u^\alpha g_{\alpha\sigma} u^\sigma. \tag{L.1}$$

For the spin-0 case as in [49], matter is represented with a scalar wave-function ψ,

$$\mathcal{L}_m = \sqrt{-g}\,\frac{1}{2}\left(\frac{\hbar^2}{m}\bar{\psi}\overleftarrow{D}_\mu D^\mu\psi - m\bar{\psi}\psi\right), \tag{L.2}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} + \frac{iQ}{\hbar}A_\mu \quad , \quad \overleftarrow{D}_\mu = \frac{\overleftarrow{\partial}}{\partial x^\mu} - \frac{iQ}{\hbar}A_\mu. \tag{L.3}$$

For the spin-1/2 case as in [49], matter is represented by a four-component wave-function ψ, and things are defined using tetrads $e_{(a)}{}^\sigma$,

$$\mathcal{L}_m = \sqrt{-g}\left(\frac{i\hbar}{2}(\bar{\psi}\gamma^\sigma D_\sigma\psi - \bar{\psi}\overleftarrow{D}_\sigma\gamma^\sigma\psi) - m\bar{\psi}\psi\right), \tag{L.4}$$

$$\gamma^\sigma = \gamma^{(a)}e_{(a)}{}^\sigma \quad , \quad \gamma^{(0)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \gamma^{(i)} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{L.5}$$

$$g^{(a)(b)} = e^{(a)}{}_\tau\, e^{(b)}{}_\sigma\, g^{\tau\sigma} = \frac{1}{2}(\gamma^{(a)}\gamma^{(b)} + \gamma^{(b)}\gamma^{(a)}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{L.6}$$

$$g^{\tau\sigma} = e_{(a)}{}^\tau e_{(b)}{}^\sigma g^{(a)(b)} = \frac{1}{2}(\gamma^\tau\gamma^\sigma + \gamma^\sigma\gamma^\tau), \tag{L.7}$$

$$e^{(a)}{}_\tau e_{(c)}{}^\tau = \delta^{(a)}_{(c)} \quad , \quad e^{(a)}{}_\tau e_{(a)}{}^\sigma = \delta^\sigma_\tau, \tag{L.8}$$

$$D_\mu = \frac{\partial}{\partial x^\mu} + \check{\Gamma}_\mu + \frac{iQ}{\hbar}A_\mu \quad , \quad \overleftarrow{D}_\mu = \frac{\overleftarrow{\partial}}{\partial x^\mu} + \check{\Gamma}^\dagger_\mu - \frac{iQ}{\hbar}A_\mu, \tag{L.9}$$

$$\check{\Gamma}_\mu = \frac{1}{2}\Sigma^{(a)(b)}e_{(a)}{}^\sigma e_{(b)\sigma;\mu} \quad , \quad \Sigma^{(a)(b)} = \frac{i}{2}(\gamma^{(a)}\gamma^{(b)} - \gamma^{(b)}\gamma^{(a)}). \tag{L.10}$$

In the equations above, m is mass, Q is charge and the σ_i are the Pauli spin matrices. In (L.3,L.9) the conjugate derivative operator $\overleftarrow{D}_\mu$ is made to operate from right to left to simplify subsequent calculations. The spin-0 and spin-1/2 $\mathcal{L}_m's$ are the ordinary expressions for quantum fields in curved space[49].

To calculate $T_{\mu\nu}$ for the spin-1/2 case we will need to first calculate the derivative $\partial(\sqrt{-g}\,e_{(a)}{}^\tau)/\partial(\sqrt{-N}N^{\dashv\mu\nu})$. Multiplying (L.7) by $\sqrt{-g}\,e^{(b)}{}_\sigma$ and taking its derivative with respect to $\sqrt{-g}\,g^{\mu\nu}$ gives

$$\sqrt{-g}\,g^{\tau\sigma}e^{(b)}{}_{\sigma} = \sqrt{-g}\,e_{(a)}{}^{\tau}g^{(a)(b)}, \tag{L.11}$$

$$\delta^{\tau}_{(\mu}\delta^{\sigma}_{\nu)}e^{(b)}{}_{\sigma} + \sqrt{-g}\,g^{\tau\sigma}\frac{\partial e^{(b)}{}_{\sigma}}{\partial(\sqrt{-g}\,g^{\mu\nu})} = \frac{\partial(\sqrt{-g}\,e_{(a)}{}^{\tau})}{\partial(\sqrt{-g}\,g^{\mu\nu})}g^{(a)(b)}. \tag{L.12}$$

Taking the derivative of (L.8) with respect to $\sqrt{-g}\,g^{\mu\nu}$ and using (2.26) gives

$$0 = e^{(a)}{}_{\tau}\frac{\partial(\sqrt{-g}\,e_{(c)}{}^{\tau})}{\partial(\sqrt{-g}\,g^{\mu\nu})} - \frac{g_{\nu\mu}}{(n-2)}\delta^{(a)}_{(c)} + \sqrt{-g}\,\frac{\partial e^{(a)}{}_{\tau}}{\partial(\sqrt{-g}\,g^{\mu\nu})}e_{(c)}{}^{\tau}. \tag{L.13}$$

Substituting (L.13) into (L.12) we finally get

$$\frac{\partial(\sqrt{-g}\,e_{(a)}{}^{\tau})}{\partial(\sqrt{-g}\,g^{\mu\nu})}g^{(a)(b)} - e^{(b)}{}_{(\nu}\delta^{\tau}_{\mu)} = \sqrt{-g}\,g^{\tau\sigma}\frac{\partial e^{(b)}{}_{\sigma}}{\partial(\sqrt{-g}\,g^{\mu\nu})} \tag{L.14}$$

$$= g^{\tau\sigma}\left(\frac{g_{\nu\mu}}{(n-2)}e^{(b)}{}_{\sigma} - e^{(b)}{}_{\tau}\frac{\partial(\sqrt{-g}\,e_{(a)}{}^{\tau})}{\partial(\sqrt{-g}\,g^{\mu\nu})}\,e^{(a)}{}_{\sigma}\right) \tag{L.15}$$

$$= \frac{g_{\nu\mu}}{(n-2)}e^{(b)\tau} - g^{(a)(b)}\frac{\partial(\sqrt{-g}\,e_{(a)}{}^{\tau})}{\partial(\sqrt{-g}\,g^{\mu\nu})}, \tag{L.16}$$

$$\frac{\partial(\sqrt{-g}\,e_{(a)}{}^{\tau})}{\partial(\sqrt{-N}N^{\dashv\mu\nu})} = \frac{\partial(\sqrt{-g}\,e_{(a)}{}^{\tau})}{\partial(\sqrt{-g}g^{\mu\nu})} = \frac{1}{2}e_{(a)(\nu}\delta^{\tau}_{\mu)} + \frac{g_{\nu\mu}}{2(n-2)}e_{(a)}{}^{\tau}. \tag{L.17}$$

From (2.30) we see that $S_{\nu\mu}$ and $T_{\nu\mu}$ are different for each $\mathcal{L}_m$ case. For the classical hydrodynamics case (L.1),

$$S_{\nu\mu} = \frac{\boldsymbol{\mu}}{\sqrt{-g}}\left(u_\nu u_\mu - \frac{1}{(n-2)}g_{\nu\mu}u^\alpha u_\alpha\right), \tag{L.18}$$

$$T_{\nu\mu} = \frac{\boldsymbol{\mu}}{\sqrt{-g}}\,u_\nu u_\mu. \tag{L.19}$$

For the spin-0 case (L.2) as in [49],

$$S_{\nu\mu} = \frac{1}{m}\left(\hbar^2\bar\psi\overleftarrow{D}_{(\nu}D_{\mu)}\psi - \frac{1}{(n-2)}g_{\nu\mu}m^2\bar\psi\psi\right), \tag{L.20}$$

$$T_{\nu\mu} = \frac{1}{m}\left(\hbar^2\bar\psi\overleftarrow{D}_{(\nu}D_{\mu)}\psi - \frac{1}{2}g_{\nu\mu}(\hbar^2\bar\psi\overleftarrow{D}_{\sigma}D^{\sigma}\psi - m^2\bar\psi\psi)\right). \tag{L.21}$$

For the spin-1/2 case (L.4) as in [49],

$$S_{\nu\mu} = \frac{i\hbar}{2}\left(\bar\psi\gamma_{(\nu}D_{\mu)}\psi - \bar\psi\overleftarrow{D}_{(\mu}\gamma_{\nu)}\psi - \frac{1}{(n-2)}g_{\nu\mu}(\bar\psi\gamma^\sigma D_\sigma\psi - \bar\psi\overleftarrow{D}_\sigma\gamma^\sigma\psi)\right), \tag{L.22}$$

$$T_{\nu\mu} = \frac{i\hbar}{2}\left(\bar\psi\gamma_{(\nu}D_{\mu)}\psi - \bar\psi\overleftarrow{D}_{(\mu}\gamma_{\nu)}\psi\right). \tag{L.23}$$

Note that in the purely classical limit as $i\hbar D_\sigma\psi\rightarrow p_\sigma\psi$, $-i\hbar\bar{\psi}\overleftarrow{D}_\sigma\rightarrow\bar{\psi}p_\sigma$, the energy-momentum tensors (L.21) for spin-0 and (L.23) for spin-1/2 both go to the classical hydrodynamics case (L.19).

From (2.46) we see that j^τ is different for each $\mathcal{L}_m$ case. For the classical hydrodynamics case (L.1),

$$j^\alpha = \frac{\boldsymbol{\mu}Q}{m\sqrt{-g}}u^\alpha. \tag{L.24}$$

For the spin-0 case (L.2) as in [49],

$$j^\alpha = \frac{i\hbar Q}{2m}(\bar{\psi}D^\alpha\psi - \bar{\psi}\overleftarrow{D}^\alpha\psi). \tag{L.25}$$

For the spin-1/2 case (L.4) as in [49],

$$j^\alpha = Q\bar{\psi}\gamma^\alpha\psi. \tag{L.26}$$

For the spin-0 case, the Klein-Gordon equation is obtained by setting $\delta\mathcal{L}/\delta\bar{\psi}=0$,

$$\begin{aligned}
0 &= \frac{-2}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \left(\frac{\partial\mathcal{L}}{\partial\bar{\psi}_{,\lambda}}\right)_{,\lambda}\right] && \text{(L.27)}\\
&= -\left[\frac{\hbar^2}{m}\left(-\frac{iQ}{\hbar}A_\mu\right)D^\mu\psi - m\psi - \frac{\hbar^2}{m\sqrt{-g}}(\sqrt{-g}D^\lambda\psi)_{,\lambda}\right] && \text{(L.28)}\\
&= \frac{-1}{m}\left[\frac{-\hbar^2}{\sqrt{-g}}\left(\frac{\partial}{\partial x^\mu} + \frac{iQ}{\hbar}A_\mu\right)\sqrt{-g}D^\mu\psi - m^2\right]\psi && \text{(L.29)}\\
&= \frac{1}{m}\left[\frac{\hbar^2}{\sqrt{-g}}D_\mu\sqrt{-g}D^\mu + m^2\right]\psi. && \text{(L.30)}
\end{aligned}$$

The conjugate Klein-Gordon equation is found by setting $\delta\mathcal{L}/\delta\psi=0$,

$$\begin{aligned}
0 &= \frac{-2}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}}{\partial\psi} - \left(\frac{\partial\mathcal{L}}{\partial\psi_{,\lambda}}\right)_{,\lambda}\right] && \text{(L.31)}\\
&= \frac{1}{m}\bar{\psi}\left[\overleftarrow{D}^\mu\sqrt{-g}\overleftarrow{D}_\mu\frac{\hbar^2}{\sqrt{-g}} + m^2\right]. && \text{(L.32)}
\end{aligned}$$

This is just the complex conjugate of the Klein-Gordon equation (L.30) if $\bar{\psi} = \psi^*$. For the spin-1/2 case, the Dirac equation is found in a similar manner,

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \left(\frac{\partial \mathcal{L}}{\partial \bar{\psi}_{,\lambda}}\right)_{,\lambda}\right] && \text{(L.33)}\\
&= i\hbar\gamma^\sigma D_\sigma \psi - m\psi. && \text{(L.34)}
\end{aligned}
$$

The conjugate Dirac equation is,

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial \psi} - \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\lambda}}\right)_{,\lambda}\right] && \text{(L.35)}\\
&= -i\hbar\bar{\psi}\overleftarrow{D}_\sigma \gamma^\sigma - m\bar{\psi}. && \text{(L.36)}
\end{aligned}
$$

Both the Klein-Gordon and Dirac equations match those of ordinary one-particle quantum mechanics in curved space[49].

Note that for the spin-0 case, instead of deriving the continuity equation (2.49,L.25) from the divergence of Ampere's law, it can also be derived from the Klein-Gordon equation. Using (L.30,L.32,L.3,L.25) we get,

$$
\begin{aligned}
0 &= \frac{iQ}{2\hbar}\left[\bar{\psi}\begin{pmatrix} one\ side\ of \\ Klein\!-\!Gordon\ equation\end{pmatrix} - \begin{pmatrix} one\ side\ of\ conjugate \\ Klein\!-\!Gordon\ equation\end{pmatrix}\psi\right] && \text{(L.37)}\\
&= \frac{iQ}{2m\hbar}\bar{\psi}\left(\frac{\hbar^2}{\sqrt{-g}}D_\mu\sqrt{-g}D^\mu + m^2 - \overleftarrow{D}^\mu\sqrt{-g}\overleftarrow{D}_\mu\frac{\hbar^2}{\sqrt{-g}} - m^2\right)\psi && \text{(L.38)}\\
&= \frac{i\hbar Q}{2m}\bar{\psi}\left(\left(\overleftarrow{D}_\mu + \frac{1}{\sqrt{-g}}D_\mu\sqrt{-g}\right)D^\mu - \overleftarrow{D}^\mu\left(\sqrt{-g}\overleftarrow{D}_\mu\frac{1}{\sqrt{-g}} + D_\mu\right)\right)\psi && \text{(L.39)}\\
&= \frac{i\hbar Q}{2m}\bar{\psi}\left(\left(\frac{\overleftarrow{\partial}}{\partial x^\mu} + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\sqrt{-g}\right)D^\mu - \overleftarrow{D}^\mu\left(\sqrt{-g}\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{1}{\sqrt{-g}} + \frac{\partial}{\partial x^\mu}\right)\right)\psi && \text{(L.40)}\\
&= \frac{i\hbar Q}{2m}\left(\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\bar{\psi}\sqrt{-g}D^\mu\psi) - (\bar{\psi}\overleftarrow{D}^\mu\sqrt{-g}\psi)\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{1}{\sqrt{-g}}\right) && \text{(L.41)}\\
&= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\frac{i\hbar Q}{2m}(\bar{\psi}D^\mu\psi - \bar{\psi}\overleftarrow{D}^\mu\psi)\right) && \text{(L.42)}\\
&= j^\mu{}_{;\mu}. && \text{(L.43)}
\end{aligned}
$$

Similarly, instead of deriving the Lorentz-force equation (4.10,L.21) from the divergence of the Einstein equations, it can also be derived from the Klein-Gordon equation. Using (L.30,L.32,L.3,L.21) we get,

$$
\begin{aligned}
0 &= \begin{pmatrix} \textit{one side of conjugate} \\ \textit{Klein-Gordon equation} \end{pmatrix} \frac{D_\rho \psi}{2} + \frac{\bar{\psi}\overleftarrow{D}_\rho}{2} \begin{pmatrix} \textit{one side of} \\ \textit{Klein-Gordon equation} \end{pmatrix} \qquad \text{(L.44)} \\
&= \bar{\psi}\left(\overleftarrow{D}^\lambda \sqrt{-g}\,\overleftarrow{D}_\lambda \frac{\hbar^2}{\sqrt{-g}} + m^2\right)\frac{D_\rho\psi}{2m} + \frac{\bar{\psi}\overleftarrow{D}_\rho}{2m}\left(\frac{\hbar^2}{\sqrt{-g}} D_\lambda \sqrt{-g} D^\lambda + m^2\right)\psi \qquad \text{(L.45)} \\
&= \frac{\hbar^2}{2m}\left[\frac{\partial(\bar{\psi}\overleftarrow{D}^\lambda\sqrt{-g})}{\partial x^\lambda}\frac{1}{\sqrt{-g}} D_\rho\psi + \bar{\psi}\overleftarrow{D}_\rho \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}D^\lambda\psi)}{\partial x^\lambda}\right. \\
&\qquad \left. +\bar{\psi}\overleftarrow{D}^\lambda\left(-\frac{iQ}{\hbar}A_\lambda\right)D_\rho\psi + \bar{\psi}\overleftarrow{D}_\rho\left(\frac{iQ}{\hbar}A_\lambda\right)D^\lambda\psi\right] \\
&\qquad + \frac{m}{2}(\bar{\psi}D_\rho\psi + \bar{\psi}\overleftarrow{D}_\rho\psi) \qquad \text{(L.46)} \\
&= \frac{\hbar^2}{2m}\left[\frac{\partial(\bar{\psi}\overleftarrow{D}^\lambda\sqrt{-g})}{\partial x^\lambda}\frac{1}{\sqrt{-g}} D_\rho\psi + \bar{\psi}\overleftarrow{D}_\rho \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}D^\lambda\psi)}{\partial x^\lambda}\right. \\
&\qquad + \bar{\psi}\overleftarrow{D}^\lambda\left(-\frac{iQ}{\hbar}A_\lambda\right)\frac{\partial\psi}{\partial x^\rho} + \bar{\psi}\overleftarrow{D}^\lambda\left(-\frac{iQ}{\hbar}A_\lambda\right)\left(\frac{iQ}{\hbar}A_\rho\right)\psi \\
&\qquad \left. +\frac{\partial\bar{\psi}}{\partial x^\rho}\left(\frac{iQ}{\hbar}A_\lambda\right)D^\lambda\psi + \left(-\frac{iQ}{\hbar}A_\rho\right)\bar{\psi}\left(\frac{iQ}{\hbar}A_\lambda\right)D^\lambda\psi\right] \\
&\qquad + \frac{m}{2}\left(\bar{\psi}\frac{\partial\psi}{\partial x^\rho} + \frac{\partial\bar{\psi}}{\partial x^\rho}\psi\right) \qquad \text{(L.47)} \\
&= \frac{\hbar^2}{2m}\left[\frac{\partial(\bar{\psi}\overleftarrow{D}^\lambda\sqrt{-g})}{\partial x^\lambda}\frac{1}{\sqrt{-g}} D_\rho\psi + \bar{\psi}\overleftarrow{D}_\rho \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}D^\lambda\psi)}{\partial x^\lambda}\right. \\
&\qquad - \bar{\psi}\overleftarrow{D}^\lambda\left(\frac{iQ}{\hbar}A_\lambda\right)\frac{\partial\psi}{\partial x^\rho} - \bar{\psi}\overleftarrow{D}^\lambda\left(-\frac{iQ}{\hbar}A_\rho\right)\frac{\partial\psi}{\partial x^\lambda} + \bar{\psi}\overleftarrow{D}^\lambda\left(-\frac{iQ}{\hbar}A_\rho\right)D^\lambda\psi \\
&\qquad \left. -\frac{\partial\bar{\psi}}{\partial x^\rho}\left(-\frac{iQ}{\hbar}A_\lambda\right)D^\lambda\psi + \left(-\frac{iQ}{\hbar}A_\rho\right)\frac{\partial\bar{\psi}}{\partial x^\lambda}D^\lambda\psi - \bar{\psi}\overleftarrow{D}^\lambda\left(-\frac{iQ}{\hbar}A_\rho\right)D^\lambda\psi\right] \\
&\qquad + \frac{m}{2}\frac{\partial(\bar{\psi}\psi)}{\partial x^\rho} \qquad \text{(L.48)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\hbar^2}{2m}\left[\frac{\partial(\bar{\psi}\overleftarrow{D}^{\lambda}\sqrt{-g})}{\partial x^{\lambda}}\frac{1}{\sqrt{-g}}D_{\rho}\psi + \bar{\psi}\overleftarrow{D}_{\rho}\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}D^{\lambda}\psi)}{\partial x^{\lambda}}\right. \\
&\quad + \bar{\psi}\overleftarrow{D}^{\lambda}\left(\frac{\partial}{\partial x^{\lambda}}\left(\frac{iQ}{\hbar}A_{\rho}\psi\right) - \frac{\partial}{\partial x^{\rho}}\left(\frac{iQ}{\hbar}A_{\lambda}\psi\right) - \frac{2iQ}{\hbar}A_{[\rho,\lambda]}\psi\right) \\
&\qquad \left. + \left(\frac{\partial}{\partial x^{\lambda}}\left(-\frac{iQ}{\hbar}A_{\rho}\bar{\psi}\right) - \frac{\partial}{\partial x^{\rho}}\left(-\frac{iQ}{\hbar}A_{\lambda}\bar{\psi}\right) + \frac{2iQ}{\hbar}A_{[\rho,\lambda]}\bar{\psi}\right)D^{\lambda}\psi\right] \\
&\quad + \frac{m}{2}\frac{\partial(\bar{\psi}\psi)}{\partial x^{\rho}} \qquad \text{(L.49)} \\
&= \frac{\hbar^2}{2m}\left[\frac{\partial(\bar{\psi}\overleftarrow{D}^{\lambda}\sqrt{-g})}{\partial x^{\lambda}}\frac{1}{\sqrt{-g}}D_{\rho}\psi + \bar{\psi}\overleftarrow{D}_{\rho}\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}D^{\lambda}\psi)}{\partial x^{\lambda}}\right. \\
&\quad + \bar{\psi}\overleftarrow{D}_{\nu}\frac{\partial g^{\nu\lambda}}{\partial x^{\rho}}D_{\lambda}\psi - \bar{\psi}\overleftarrow{D}_{\mu}\frac{\partial g^{\mu\lambda}}{\partial x^{\rho}}D_{\lambda}\psi \\
&\quad + \bar{\psi}\overleftarrow{D}^{\lambda}\left(\frac{\partial(D_{\rho}\psi)}{\partial x^{\lambda}} - \frac{\partial(D_{\lambda}\psi)}{\partial x^{\rho}}\right) + \left(\frac{\partial(\bar{\psi}\overleftarrow{D}_{\rho})}{\partial x^{\lambda}} - \frac{\partial(\bar{\psi}\overleftarrow{D}_{\lambda})}{\partial x^{\rho}}\right)D^{\lambda}\psi \\
&\quad \left. + \frac{2iQ}{\hbar}(\bar{\psi}D^{\lambda}\psi - \bar{\psi}\overleftarrow{D}^{\lambda}\psi)A_{[\rho,\lambda]}\right] + \frac{m}{2}\frac{\partial(\bar{\psi}\psi)}{\partial x^{\rho}} \qquad \text{(L.50)} \\
&= \frac{\hbar^2}{2m}\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\lambda}}\Big(\sqrt{-g}(\bar{\psi}\overleftarrow{D}^{\lambda}D_{\rho}\psi + \bar{\psi}\overleftarrow{D}_{\rho}D^{\lambda}\psi)\Big) + \bar{\psi}\overleftarrow{D}_{\lambda}\frac{\partial g^{\lambda\nu}}{\partial x^{\rho}}D_{\nu}\psi - \frac{\partial}{\partial x^{\rho}}\bar{\psi}\overleftarrow{D}_{\lambda}D^{\lambda}\psi\right] \\
&\quad + \frac{m}{2}\frac{\partial(\bar{\psi}\psi)}{\partial x^{\rho}} + \frac{iQ\hbar}{m}(\bar{\psi}D^{\lambda}\psi - \bar{\psi}\overleftarrow{D}^{\lambda}\psi)A_{[\rho,\lambda]} \qquad \text{(L.51)} \\
&= \frac{1}{2m}\left[\hbar^2(\bar{\psi}\overleftarrow{D}^{\lambda}D_{\rho}\psi + \bar{\psi}\overleftarrow{D}_{\rho}D^{\lambda}\psi)_{;\lambda} - (\hbar^2\bar{\psi}\overleftarrow{D}_{\lambda}D^{\lambda}\psi - m^2\bar{\psi}\psi)_{,\rho}\right] \\
&\quad + \frac{iQ\hbar}{m}(\bar{\psi}D^{\lambda}\psi - \bar{\psi}\overleftarrow{D}^{\lambda}\psi)A_{[\rho,\lambda]} \qquad \text{(L.52)} \\
&= T^{\lambda}_{\rho;\,\lambda} + 2j^{\lambda}A_{[\rho,\lambda]}. \qquad \text{(L.53)}
\end{aligned}
$$

Presumably, similar results occur for the spin-1/2 case, but this was not verified.

Appendix M

Alternative ways to derive the Einstein-Schrödinger theory

The original Einstein-Schrödinger theory can be derived from many different Lagrangian densities. In fact it results from any Lagrangian density of the form,

$$\mathcal{L}(\widehat{\Gamma}^{\lambda}_{\rho\tau}, N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}(R_{\nu\mu}(\tilde{\Gamma})+c_1\tilde{\Gamma}^{\alpha}_{\alpha[\nu,\mu]}+2A_{[\nu,\mu]}\sqrt{2}\,i\Lambda_b^{1/2}) + (n-2)\Lambda_b\right], \quad (M.1)$$

where c_1, c_2, c_3 are arbitrary constants and

$$R_{\nu\mu}(\tilde{\Gamma}) = \tilde{\Gamma}^{\alpha}_{\nu\mu,\alpha} - \tilde{\Gamma}^{\alpha}_{\nu\alpha,\mu} + \tilde{\Gamma}^{\sigma}_{\nu\mu}\tilde{\Gamma}^{\alpha}_{\sigma\alpha} - \tilde{\Gamma}^{\sigma}_{\nu\alpha}\tilde{\Gamma}^{\alpha}_{\sigma\mu}, \quad (M.2)$$

$$\tilde{\Gamma}^{\alpha}_{\nu\mu} = \widehat{\Gamma}^{\alpha}_{\nu\mu} + \frac{2}{(n-1)}\left[c_2\delta^{\alpha}_{\mu}\widehat{\Gamma}^{\sigma}_{[\sigma\nu]} + (c_2-1)\,\delta^{\alpha}_{\nu}\widehat{\Gamma}^{\sigma}_{[\sigma\mu]}\right], \quad (M.3)$$

$$A_{\nu} = \widehat{\Gamma}^{\sigma}_{[\sigma\nu]}/c_3. \quad (M.4)$$

Contracting (M.3) on the right and left gives

$$\tilde{\Gamma}^{\alpha}_{\beta\alpha} = \frac{1}{(n-1)}\left[(c_2 n + c_2 - 1)\widehat{\Gamma}^{\alpha}_{\alpha\beta} - (c_2 n + c_2 - n)\widehat{\Gamma}^{\alpha}_{\beta\alpha}\right] = \tilde{\Gamma}^{\alpha}_{\alpha\beta}, \quad (M.5)$$

so $\tilde{\Gamma}^\alpha_{\nu\mu}$ has only $n^3 - n$ independent components. Also, from (M.3,M.4) we have

$$\widehat{\Gamma}^\alpha_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu} - \frac{2c_3}{(n-1)}\left[c_2\delta^\alpha_\mu A_\nu + (c_2-1)\delta^\alpha_\nu A_\mu\right], \quad \text{(M.6)}$$

so $\tilde{\Gamma}^\alpha_{\nu\mu}$ and A_ν fully parameterize $\widehat{\Gamma}^\alpha_{\nu\mu}$ and can be treated as independent variables. Therefore setting $\delta\mathcal{L}/\delta\tilde{\Gamma}^\alpha_{\nu\mu} = 0$ and $\delta\mathcal{L}/\delta A_\nu = 0$ must give the same field equations as $\delta\mathcal{L}/\delta\widehat{\Gamma}^\alpha_{\nu\mu} = 0$. Because the field equations can be derived in this way, the constants c_2 and c_3 are clearly arbitrary, and because of (2.58) with $j^\sigma = 0$, the constant c_1 is also arbitrary.

For $c_1 = 1$, $c_2 = 1/2$, $c_3 = -(n-1)\sqrt{2}\, i\Lambda_b^{1/2}$, (M.1) reduces to

$$\mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau}, N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) + (n-2)\Lambda_b\right], \quad \text{(M.7)}$$

which is our original Lagrangian density (2.2) without the $\Lambda_z\sqrt{-g}$ and $\mathcal{L}_m$ terms, where we have the invariance properties from (2.18,2.19),

$$A_\nu \to -A_\nu,\ \tilde{\Gamma}^\alpha_{\nu\mu} \to \tilde{\Gamma}^\alpha_{\mu\nu},\ \widehat{\Gamma}^\alpha_{\nu\mu} \to \widehat{\Gamma}^\alpha_{\mu\nu},\ N_{\nu\mu} \to N_{\mu\nu},\ N^{\dashv\mu\nu} \to N^{\dashv\nu\mu} \ \Rightarrow\ \mathcal{L} \to \mathcal{L}, \quad \text{(M.8)}$$

$$A_\alpha \to A_\alpha - \frac{\hbar}{Q}\phi_{,\alpha},\ \tilde{\Gamma}^\alpha_{\rho\tau} \to \tilde{\Gamma}^\alpha_{\rho\tau},\ \widehat{\Gamma}^\alpha_{\rho\tau} \to \widehat{\Gamma}^\alpha_{\rho\tau} + \frac{2\hbar}{Q}\delta^\alpha_{[\rho}\phi_{,\tau]}\sqrt{2}\, i\Lambda_b^{1/2} \ \Rightarrow\ \mathcal{L} \to \mathcal{L}. \quad \text{(M.9)}$$

For this case we have $\tilde{\Gamma}^\alpha_{\sigma\alpha} = \tilde{\Gamma}^\alpha_{\alpha\sigma} = \widehat{\Gamma}^\alpha_{(\alpha\sigma)}$, and from (2.57,2.28,M.1) the field equations require a generalization of the result $\mathcal{L}_{,\sigma} - \Gamma^\alpha_{\alpha\sigma}\mathcal{L} = 0$ that occurs with the Lagrangian density of ordinary vacuum general relativity, that is

$$\mathcal{L}_{,\sigma} - \widehat{\Gamma}^\alpha_{(\alpha\sigma)}\mathcal{L} = 0 \quad \text{or} \quad \mathcal{L}_{,\sigma} - Re(\widehat{\Gamma}^\alpha_{\alpha\sigma})\mathcal{L} = 0. \quad \text{(M.10)}$$

For the alternative choice, $c_1 = 0$, $c_2 = n/(n+1)$, $c_3 = -(n-1)\sqrt{2}\, i\Lambda_b^{1/2}/2$, we have $\tilde{\Gamma}^\alpha_{\sigma\alpha} = \tilde{\Gamma}^\alpha_{\alpha\sigma} = \widehat{\Gamma}^\alpha_{\alpha\sigma}$ and from (2.57,2.28,M.1) the field equations require

$$\mathcal{L}_{,\sigma} - \widehat{\Gamma}^\alpha_{\alpha\sigma}\mathcal{L} = 0. \quad \text{(M.11)}$$

For the alternative choice $c_1\!=\!1,\ c_2\!=\!0,\ c_3\!=\!-(n\!-\!1)\sqrt{2}\,i\Lambda_b^{1/2}$, (M.1) reduces to

$$\mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau},N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}\Re_{\nu\mu}(\widehat{\Gamma}) + (n\!-\!2)\Lambda_b\right], \tag{M.12}$$

where $\Re_{\nu\mu}(\widehat{\Gamma})$ is a fairly simple generalization of the ordinary Ricci tensor

$$\Re_{\nu\mu}(\widehat{\Gamma}) \;=\; \widehat{\Gamma}^\alpha_{\nu\mu,\alpha} - \widehat{\Gamma}^\alpha_{(\nu|\alpha,|\mu)} + \widehat{\Gamma}^\sigma_{\nu\mu}\widehat{\Gamma}^\alpha_{\sigma\alpha} - \widehat{\Gamma}^\sigma_{\nu\alpha}\widehat{\Gamma}^\alpha_{\sigma\mu}. \tag{M.13}$$

For the alternative choice $c_1\!=\!0,\ c_2\!=\!0,\ c_3\!=\!-(n\!-\!1)\sqrt{2}\,i\Lambda_b^{1/2}/2$, (M.1) reduces to

$$\mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau},N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}R_{\nu\mu}(\widehat{\Gamma}) + (n\!-\!2)\Lambda_b\right], \tag{M.14}$$

where $R_{\nu\mu}(\widehat{\Gamma})$ is the ordinary Ricci tensor

$$R_{\nu\mu}(\widehat{\Gamma}) \;=\; \widehat{\Gamma}^\alpha_{\nu\mu,\alpha} - \widehat{\Gamma}^\alpha_{\nu\alpha,\mu} + \widehat{\Gamma}^\sigma_{\nu\mu}\widehat{\Gamma}^\alpha_{\sigma\alpha} - \widehat{\Gamma}^\sigma_{\nu\alpha}\widehat{\Gamma}^\alpha_{\sigma\mu}. \tag{M.15}$$

The original Einstein-Schrödinger theory (including the cosmological constant) can even be derived from purely affine versions of the Lagrangian densities described above, such as the Lagrangian density $\mathcal{L}(\widehat{\Gamma}) = [-det(R_{\nu\mu}(\widehat{\Gamma}))]^{1/2}$ used by Schrödinger[6]. A better choice is the purely affine version of (M.7)

$$\mathcal{L}(\widehat{\Gamma}) = \frac{\Lambda_b}{16\pi}\sqrt{-det(N_{\nu\mu})}, \tag{M.16}$$

where $N_{\nu\mu}$ is simply defined to be

$$N_{\nu\mu} = -\mathcal{R}_{\nu\mu}(\widehat{\Gamma})/\Lambda_b, \tag{M.17}$$

and the properties (M.8,M.9,M.10) are inherited. Decomposing $\widehat{\Gamma}^\alpha_{\nu\mu}$ into $\tilde{\Gamma}^\alpha_{\nu\mu}$ and A_σ as in (2.4,2.6,2.7), and using (2.9,2.11) it is simple to show that $\delta\mathcal{L}/\delta A_\nu = 0$ and $\delta\mathcal{L}/\delta\tilde{\Gamma}^\alpha_{\nu\mu}\!=0$ give identical equations as in §2.3 and §2.4 except that $\mathcal{L}_m\!=\!0,\ j_\mu\!=\!0,$

$T_{\mu\nu}=0$, $\Lambda_z=0$. In addition, the definition (M.17) matches the field equations (2.28), so that this equation and all of the subsequent equations in §2.2 are identical except that $\mathcal{L}_m=0$, $j_\mu=0$, $T_{\mu\nu}=0$, $\Lambda_z=0$. Therefore, the purely affine Lagrangian density (M.16,M.17,2.5) gives the same theory as the Palatini Lagrangian density (2.2) with $\mathcal{L}_m=0$, $\Lambda_z=0$, which is the original Einstein-Schrödinger theory. The derivation of the Einstein-Schrödinger theory in this manner is remarkable because the only fundamental field assumed *a priori* is the connection $\widehat{\Gamma}^\alpha_{\sigma\mu}$. The fields $N_{\sigma\mu}$, $g_{\sigma\mu}$, $f_{\sigma\mu}$, A_σ and $\tilde{\Gamma}^\alpha_{\sigma\mu}$ all just appear as convenient variables to work with when solving the field equations.

It is important to note that the purely affine derivation only works for Schrödinger's generalization of Einstein's theory which includes a bare cosmological constant, because if $\Lambda_b=0$, the definition (M.17) would not make sense. Also note that the only reason we do not set $\Lambda_b=1$ is because we are assuming the convention that $N_{\sigma\mu}$ has values close to 1. If we chose to we would be free to absorb Λ_b into $N_{\sigma\mu}$ because both $\tilde{\Gamma}^\alpha_{\sigma\mu}(N_{..})$ and $R_{\sigma\mu}(\tilde{\Gamma}(N_{..}))$ are independent of a constant multiplier on $N_{\sigma\mu}$. We would also be free to absorb Λ_b into the definition of A_σ. Therefore, Λ_b does not need to be in either the field equations or the Lagrangian density. It is only there to make the definitions of $N_{\sigma\mu}$ and A_σ conform to conventions. The cosmological constant term has often been referred to as an undesirable complication, attached to otherwise elegant field equations to make them conform to reality. From the standpoint of the derivation above, it is nothing of the sort. Instead, Λ_b appears as the magnitude of the fundamental tensor $N_{\sigma\mu}$ when $N_{\sigma\mu}$ is put in more natural units. The cosmological constant term is not an added-on appendage to this theory but is instead an inherent part of it.

Let us consider whether the Lagrangian density (M.16,M.17,2.5) is unique in that the resulting theory satisfies (M.10). While a rigorous proof is probably not possible, a strong argument will be presented below that the theory is unique in this property. With no metric to use, the forms that a scalar density can take are limited. Also, because (M.10) exists for any dimension, we must only consider forms which exist for any dimension. To discuss this topic, it is convenient to use the fields $\tilde{\Gamma}^{\alpha}_{\sigma\mu}, A_\sigma$ as defined by (2.7,2.4) instead of $\widehat{\Gamma}^{\alpha}_{\sigma\mu}$. The simplest form to consider is $\mathcal{L} = \sqrt{-N}$, where $N_{\sigma\mu}$ is a linear combination of the terms $\tilde{\mathcal{R}}_{\sigma\mu}$, $\tilde{\mathcal{R}}_{\mu\sigma}$, $\tilde{\Gamma}^{\alpha}_{\alpha[\mu,\sigma]}$, $A_{[\sigma,\mu]}$, $A_{\sigma,\mu} - \tilde{\Gamma}^{\alpha}_{\sigma\mu}A_\alpha$, $\tilde{\Gamma}^{\alpha}_{[\sigma\mu]}A_\alpha$, and $A_\sigma A_\mu$. Many other terms can be decomposed into these, such as $R_{\sigma\mu}(\tilde{\Gamma}^T) = \tilde{\mathcal{R}}_{\mu\sigma} + 2\tilde{\Gamma}^{\alpha}_{\alpha[\mu,\sigma]}$, $\tilde{\mathcal{R}}^{\alpha}{}_{\alpha\sigma\mu} = 2\tilde{\Gamma}^{\alpha}_{\alpha[\mu,\sigma]}$, and anything dependent on $\widehat{\Gamma}^{\alpha}_{\sigma\mu}$. Our Lagrangian density (M.16) is a special case of this form. In fact, it happens that (M.10) is satisfied for any $\mathcal{L} = \sqrt{-N}$ where $N_{\sigma\mu} = a\tilde{\mathcal{R}}_{\sigma\mu} + bA_{[\sigma,\mu]} + c\tilde{\Gamma}^{\alpha}_{\alpha[\mu,\sigma]}$ and $a \neq 0, b \neq 0$. This would initially seem to indicate that the Einstein-Schrödinger theory is not unique, except for the surprising fact that the same field equations result regardless of the coefficients in the linear combination. The $\tilde{\Gamma}^{\alpha}_{\alpha[\mu,\sigma]}$ term causes $\delta^{\rho}_{\beta}(\sqrt{-N}\, N^{\dashv[\tau\omega]})_{,\omega}$ terms in the $\delta\mathcal{L}/\delta\tilde{\Gamma}^{\beta}_{\tau\rho} = 0$ field equations (2.55), but these are required to vanish by the $\delta\mathcal{L}/\delta A_\tau = 0$ field equations (2.45). Also, (M.10) requires that $\tilde{\Gamma}^{\alpha}_{\alpha[\mu,\sigma]} = (ln\mathcal{L})_{,[\mu,\sigma]} = 0$ from (2.58), so this term is of no consequence. Different field equations result if any other terms are included in $N_{\sigma\mu}$, but then (M.10) is no longer satisfied. To argue the case for uniqueness, we must next consider more complicated forms. The most obvious generalization of a single $\sqrt{-N}$ consists of linear combinations of such terms, $\sqrt{-{}^1N}$ and $\sqrt{-{}^2N}$ etcetera. The resulting field equations contain different $N^{\dashv\sigma\mu}$ terms, and there is just no way to contract the equations to remove these terms as we did in (2.57). Linear combinations of terms such as $\sqrt{-{}^1N}\sqrt{-{}^2N}/\sqrt{-{}^3N}$ have the same characteristic. Next one can include linear combinations of terms like $\sqrt{-{}^1N}\,{}^1N^{\dashv\sigma\mu}\,{}^2N_{\mu\sigma}$. In this case the field equations contain terms with different powers of ${}^1N^{\dashv\sigma\mu}$. From trying a few of these, it seems very likely that the simplicity of (M.10) demands simplicity in the Lagrangian density, and that the only real prospect is a single $\sqrt{-N}$ as we considered originally.

Whether one prefers the Lagrangian density (M.16,M.17,2.5) with the properties (M.8-M.10) or one of the alternatives, it is clear that the original Einstein-Schrödinger theory can be derived from rather simple principles. The theory proposed in this paper just adds a $\Lambda_z\sqrt{-g}$ term to the original Einstein-Schrödinger theory, and this could be caused by zero-point fluctuations. One might perhaps regard a spin-1/2 $\mathcal{L}_m$ term (L.4) as another quantization effect, that is as the "first quantization" of our charged solution. In this case all of one-particle quantum electrodynamics results by including quantization effects in the original Einstein-Schrödinger theory. Furthermore, if one was to try to second quantize the theory, the most obvious approach would be to use path integral methods with the action of the original Einstein-Schrödinger theory, $S=\int\mathcal{L}dx^0..dx^n$ with (M.16,M.17,2.5). Since both Λ_z and a spin-1/2 $\mathcal{L}_m$ term can be interpreted as quantization effects, these terms might be expected to result as quantization effects using a purely classical action, and adding up the $e^{iS/\hbar}$ amplitudes for all "paths" of the field $\widehat{\Gamma}^\alpha_{\mu\nu}$. Now it is unclear whether such a quantization scheme would work, or how practical it would be in terms of being able to do the calculations and predict experimental results. However, it is at least theoretically possible.

The search for simple principles has led to many advances in physics, and is what led Einstein to general relativity and also to the Einstein-Schrödinger theory[87, 3]. Einstein disliked the term $\sqrt{-g}F^{\nu\mu}F_{\mu\nu}/16\pi$ in the Einstein-Maxwell Lagrangian density. Referring to the equation $G_{\nu\mu} = 8\pi T_{\nu\mu}$ he states[87] "The right side is a formal condensation of all things whose comprehension in the sense of a field-theory is still problematic. Not for a moment, of course, did I doubt that this formulation was merely a makeshift in order to give the general principle of relativity a preliminary closed expression. For it was essentially not anything more than a theory of the gravitational field, which was somewhat artificially isolated from a total field of as yet unknown structure." In modern times the term $\sqrt{-g}F^{\nu\mu}F_{\mu\nu}/16\pi$ has become standard and is rarely questioned. The theory presented here suggests that this term should be questioned, and offers an alternative which is based on simple principles and which genuinely unifies gravitation and electromagnetism.

Appendix N

Derivation of the electric monopole solution

Here we derive the exact charged solution (3.1-3.4) discussed in §3.1. It can be shown[46] that the assumption of spherical symmetry allows the fundamental tensor to be written in the following form

$$N_{\nu\mu} = \begin{pmatrix} \gamma & -w & 0 & 0 \\ w & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & r^2 v \sin\theta \\ 0 & 0 & -r^2 v \sin\theta & -\beta \sin^2\theta \end{pmatrix}. \tag{N.1}$$

Both [46] and [47] assume this form with $\beta = r^2, v = 0$ to derive a solution to the original Einstein-Schrödinger field equations which looks similar to a charged mass, but with some problems. Here we will derive a solution to the modified field equations (2.31-2.8) which is much closer to the Reissner-Nordström solution[61, 62] of electro-vac Einstein-Maxwell theory. We will follow a similar procedure to [46, 47] but will use coordinates $x_0, x_1, x_2, x_3 = ct, r, \theta, \phi$ instead of $x_1, x_2, x_3, x_4 = r, \theta, \phi, ct$. We also use the variables $a = 1/\alpha,\ b = \gamma\alpha,\ \check{s} = -w$, which allow a simpler solution than the variables α, γ, w. This gives

$$N_{\nu\mu} = \begin{pmatrix} ab & \check{s} & 0 & 0 \\ -\check{s} & -1/a & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \tag{N.2}$$

$$N^{\dashv\mu\nu} = \begin{pmatrix} 1/ad & \check{s}/d & 0 & 0 \\ -\check{s}/d & -ab/d & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}, \tag{N.3}$$

$$\sqrt{-N} = \sqrt{d}\, r^2 \sin\theta, \tag{N.4}$$

where

$$d = b - \check{s}^2. \tag{N.5}$$

From (N.3,N.4) and the definitions (2.4,2.22) of $g_{\nu\mu}$ and $f_{\nu\mu}$ we get

$$g^{\nu\mu} = \frac{1}{\check{c}}\begin{pmatrix} 1/ad & 0 & 0 & 0 \\ 0 & -ab/d & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}, \quad f^{\nu\mu} = \frac{\Lambda_b^{1/2}}{\sqrt{2}\,i\,\check{c}}\begin{pmatrix} 0 & -\check{s}/d & 0 & 0 \\ \check{s}/d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{N.6}$$

$$g_{\nu\mu} = \check{c}\begin{pmatrix} ad & 0 & 0 & 0 \\ 0 & -d/ab & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \quad f_{\nu\mu} = \frac{\Lambda_b^{1/2}}{\sqrt{2}\,i\,\check{c}}\begin{pmatrix} 0 & \check{s} & 0 & 0 \\ -\check{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{N.7}$$

$$\sqrt{-g} = \sqrt{b}\,r^2 \sin\,\theta, \tag{N.8}$$

where

$$\check{c} = \sqrt{b/d} = \sqrt{-g}/\sqrt{-N}\,. \tag{N.9}$$

Using prime ($'$) to represent $\partial/\partial r$, Ampere's law (2.47) and (N.3,N.4) require that

$$0 = (\sqrt{-N}N^{\dashv[01]})_{,1} = \left(\frac{\check{s}r^2 sin\,\theta}{\sqrt{d}}\right)'. \tag{N.10}$$

From (N.10,N.5), this means that for some constant Q we have

$$\frac{\check{s}r^2}{\sqrt{d}} = \frac{\check{s}r^2}{\sqrt{b-\check{s}^2}} = \frac{Q\sqrt{2}\,i}{\Lambda_b^{1/2}}. \tag{N.11}$$

Solving this for $\check{s}^2$ gives

$$\check{s}^2 = \frac{2bQ^2}{2Q^2 - \Lambda_b r^4}\,. \tag{N.12}$$

From (N.11,N.12) we can derive the useful relationship

$$\check{s}' = \frac{(\check{s}^2)'}{2\check{s}} = \frac{1}{2\check{s}}\left(\frac{2b'Q^2}{2Q^2 - \Lambda_b r^4} + \frac{8b\Lambda_b r^3 Q^2}{2Q^2 - \Lambda_b r^4}\left(\frac{\check{s}^2}{2bQ^2}\right)\right) = \frac{\check{s}}{b}\left(\frac{b'}{2} - \frac{2d}{r}\right). \tag{N.13}$$

The connection equations (2.55) are solved in [46, 47]. In terms of our variables, the non-zero connections are

$$\tilde\Gamma^1_{00}=\frac{a}{2}(ab)'+\frac{4a^2\check s^2}{r}\ \ ,\ \tilde\Gamma^0_{10}=\tilde\Gamma^0_{01}=\frac{(ab)'}{2ab}+\frac{2\check s^2}{br}\ \ ,\ \tilde\Gamma^1_{11}=\frac{-a'}{2a},$$
$$\tilde\Gamma^2_{12}=\tilde\Gamma^2_{21}=\tilde\Gamma^3_{13}=\tilde\Gamma^3_{31}=\frac{1}{r}, \tag{N.14}$$
$$\tilde\Gamma^1_{22}=-ar\ \ ,\ \tilde\Gamma^1_{33}=-ar\sin^2\theta\ \ ,\ \tilde\Gamma^3_{23}=\tilde\Gamma^3_{32}=\cot\theta\ \ ,\ \tilde\Gamma^2_{33}=-\sin\theta\cos\theta,$$
$$\tilde\Gamma^2_{02}=-\tilde\Gamma^2_{20}=\tilde\Gamma^3_{03}=-\tilde\Gamma^3_{30}=-\frac{a\check s}{r}\ \ ,\ \tilde\Gamma^1_{10}=-\tilde\Gamma^1_{01}=-\frac{2a\check s}{r},$$
$$\tilde\Gamma^\alpha_{\alpha 0}=0,\ \ \tilde\Gamma^\alpha_{\alpha 1}=\frac{b'}{2b}+\frac{2\check s^2}{br}+\frac{2}{r},\ \ \tilde\Gamma^\alpha_{\alpha 2}=\cot\theta,\ \ \tilde\Gamma^\alpha_{\alpha 3}=0. \tag{N.15}$$

The Ricci tensor is also calculated in [46, 47]. From (N.15) we have $\tilde\Gamma^\alpha_{\alpha[\nu,\mu]}=0$ as expected from (2.58), and this means that $\tilde{\mathcal{R}}_{\nu\mu}=\tilde R_{\nu\mu}$. In terms of our variables, and using our own sign convention, the non-zero components of the Ricci tensor are

$$\begin{aligned}
-\tilde{\mathcal{R}}_{00} &= -\frac{aba''}{2}-\frac{a^2b''}{2}-\frac{3aa'b'}{4}+\frac{a^2b'b'}{4b}-\frac{a}{r}(ab'+a'b)-\frac{8a^2\check s\check s'}{r}\\
&\quad+\frac{a^2\check s^2}{r}\left(\frac{3b'}{b}-\frac{3a'}{a}-\frac{10}{r}+\frac{8\check s^2}{br}\right), && \text{(N.16)}\\
-\tilde{\mathcal{R}}_{11} &= \frac{a''}{2a}+\frac{b''}{2b}-\frac{b'b'}{4b^2}+\frac{3a'b'}{4ab}+\frac{a'}{ar}+\frac{4\check s\check s'}{br}+\frac{\check s^2}{br}\left(\frac{3a'}{a}+\frac{4\check s^2}{br}-\frac{2}{r}\right), && \text{(N.17)}\\
-\tilde{\mathcal{R}}_{22} &= \frac{ar}{2}\left(\frac{2a'}{a}+\frac{b'}{b}\right)+a-1+\frac{2a\check s^2}{b}, && \text{(N.18)}\\
-\tilde{\mathcal{R}}_{33} &= -\tilde{\mathcal{R}}_{22}\sin^2\theta, && \text{(N.19)}\\
-\tilde{\mathcal{R}}_{[10]} &= 2\left(\frac{a\check s}{r}\right)'+\frac{6a\check s}{r^2}.\quad \{[46]\ \textit{has an error here}\} && \text{(N.20)}
\end{aligned}$$

From (N.2,N.7,N.9,N.19), the symmetric part of the field equations (2.31) is

$$
\begin{aligned}
0 &= \tilde{\mathcal{R}}_{00} + \Lambda_b N_{00} + \Lambda_z g_{00} = \tilde{\mathcal{R}}_{00} + \Lambda_b ab + \Lambda_z \frac{ab}{\check{c}} \ , &\text{(N.21)}\\
0 &= \tilde{\mathcal{R}}_{11} + \Lambda_b N_{11} + \Lambda_z g_{11} = \tilde{\mathcal{R}}_{11} - \Lambda_b \frac{1}{a} - \Lambda_z \frac{1}{a\check{c}} \ , &\text{(N.22)}\\
0 &= \tilde{\mathcal{R}}_{22} + \Lambda_b N_{22} + \Lambda_z g_{22} = \tilde{\mathcal{R}}_{22} - \Lambda_b r^2 - \Lambda_z \check{c} r^2, &\text{(N.23)}\\
0 &= \tilde{\mathcal{R}}_{33} + \Lambda_b N_{33} + \Lambda_z g_{33} = (\tilde{\mathcal{R}}_{22} + \Lambda_b N_{22} + \Lambda_z g_{22}) \sin^2\theta. &\text{(N.24)}
\end{aligned}
$$

Forming a linear combination of (N.22,N.21) and using (N.17,N.16,N.13,N.5), we find that many of the terms cancel initially and we get,

$$
\begin{aligned}
0 &= b\left(-\tilde{\mathcal{R}}_{11} + \Lambda_b \frac{1}{a} + \Lambda_z \frac{1}{a\check{c}}\right) + \frac{1}{a^2}\left(-\tilde{\mathcal{R}}_{00} - \Lambda_b ab - \Lambda_z \frac{ab}{\check{c}}\right) &\text{(N.25)}\\
&= \frac{4\check{s}\check{s}'}{r} + \frac{\check{s}^2}{r}\left(\frac{4\check{s}^2}{br} - \frac{2}{r}\right) - \frac{b'}{r} - \frac{8\check{s}\check{s}'}{r} + \frac{\check{s}^2}{r}\left(\frac{3b'}{b} - \frac{10}{r} + \frac{8\check{s}^2}{br}\right)\\
&= -\frac{4\check{s}}{r}\left[\frac{\check{s}}{b}\left(\frac{b'}{2} - \frac{2d}{r}\right)\right] + \frac{12\check{s}^2}{r}\left(\frac{\check{s}^2}{br} - \frac{1}{r}\right) - \frac{b'}{r} + \frac{3\check{s}^2 b'}{br}\\
&= -\frac{d}{br^2}\left(4\check{s}^2 + rb'\right). &\text{(N.26)}
\end{aligned}
$$

From (N.12) this requires

$$
0 = \frac{8bQ^2}{2Q^2 - \Lambda_b r^4} + rb'. \qquad \text{(N.27)}
$$

Solving (N.27) and using (N.12,N.5,N.9) gives identical results to [46, 47],

$$
\begin{aligned}
b &= 1 - \frac{2Q^2}{\Lambda_b r^4}, &\text{(N.28)}\\
\check{s} &= \sqrt{\frac{2bQ^2}{2Q^2 - \Lambda_b r^4}} = \frac{\sqrt{2}\, iQ}{\sqrt{\Lambda_b}\, r^2}, &\text{(N.29)}\\
d &= b - \check{s}^2 = 1, &\text{(N.30)}\\
\check{c} &= \sqrt{b/d} = \sqrt{1 - \frac{2Q^2}{\Lambda_b r^4}}\,. &\text{(N.31)}
\end{aligned}
$$

To find the variable "a", the 22 component of the field equations will be used. The solution is guessed to be that of [46, 47] plus an extra term $-\Lambda_z V/r$,

$$a = 1 - \frac{2M}{r} - \frac{\Lambda_b r^2}{3} - \frac{\Lambda_z V}{r}. \tag{N.32}$$

Because "b" and "$\check{s}$" are the same as [46, 47], we just need to look at the extra terms that result from Λ_z. Using (N.23,N.18,N.32,N.26,N.31) gives,

$$0 = -\tilde{\mathcal{R}}_{22} + \Lambda_b r^2 + \Lambda_z \check{c} r^2 = \frac{ar}{2}\left(\frac{2a'}{a} + \frac{b'}{b}\right) + a - 1 + \frac{2a\check{s}^2}{b} + \Lambda_b r^2 + \Lambda_z \check{c} r^2 \tag{N.33}$$

$$= -\Lambda_z \left[r\left(\frac{V}{r}\right)' + \frac{Vb'}{2b} + \frac{V}{r} + \frac{2V\check{s}^2}{rb} - \check{c}r^2 \right] = -\Lambda_z \left[V' - r^2\check{c} \right]. \tag{N.34}$$

This same equation is also obtained if the 11 or 00 components of the field equations are used. The solution for $V(r)$ can be written in terms of an elliptic integral but we will not need to calculate it. With (N.34) and the definition

$$\hat{V} = \frac{r\Lambda_b}{Q^2}\left(V - \frac{r^3}{3}\right) \tag{N.35}$$

we get the following results which will be used shortly,

$$\hat{V}' = \frac{\hat{V}}{r} + \frac{r^3\Lambda_b(\check{c}-1)}{Q^2}, \qquad \frac{Q^2}{\Lambda_b r}\left(\frac{\hat{V}}{r^2}\right)' = \check{c} - 1 - \frac{Q^2\hat{V}}{\Lambda_b r^4}. \tag{N.36}$$

Next we consider the antisymmetric part of the field equations (2.32), where only the 10 component is non-vanishing. Using (N.20,N.2,N.29,N.32) gives

$$F_{01} = \frac{\Lambda_b^{-1/2}}{\sqrt{2}\,i}(\tilde{\mathcal{R}}_{[01]} + \Lambda_b N_{[01]}) = \frac{\Lambda_b^{-1/2}}{\sqrt{2}\,i}\left[2\left(\frac{a\check{s}}{r}\right)' + \frac{6a\check{s}}{r^2} + \Lambda_b\check{s}\right] \tag{N.37}$$

$$= 2\left(\frac{aQ}{\Lambda_b r^3}\right)' + \frac{6aQ}{\Lambda_b r^4} + \frac{Q}{r^2} = \frac{Q}{r^2}\left(1 + \frac{2a'}{\Lambda_b r}\right). \tag{N.38}$$

Using (N.7,N.28,N.29,N.30,N.31,N.32,N.34,N.38,N.35,N.36) we can put the solution in its final form (3.1-3.4).

Appendix O

The function $\hat{V}(r)$ in the electric monopole solution

The behavior of the charged solution of LRES theory near the origin hinges on the behavior of the function $\hat{V}(r)$ from (3.6,3.5). The Reissner-Nordström solution has a naked singularity for $Q > M$, and this is commonly cited as a reason that this solution should not be associated with an elementary charge. Therefore it is important to investigate the behavior of $\hat{V}(r)$ near the origin, to understand how our solution differs from the Reissner-Nordström solution.

Changing variables with $t = r/r_0$ where $r_0 = \sqrt{Q}(2/\Lambda_b)^{1/4}$ from (3.16) gives

$$\hat{V} = \frac{2r}{r_0^4}\left(V - \frac{r^3}{3}\right), \qquad V = \int \sqrt{r^4 - r_0^4}\, dr = r_0^3 \int \sqrt{t^4 - 1}\; dt. \tag{O.1}$$

Integration by parts gives

$$\begin{aligned}
\int \sqrt{t^4-1}\,dt &= \frac{2}{3}\int \sqrt{t^4-1}\,dt + \frac{1}{3}\left[t\sqrt{t^4-1} - \int \frac{2t^4 dt}{\sqrt{t^4-1}}\right] && \text{(O.2)}\\
&= \frac{1}{3}\left[t\sqrt{t^4-1} + 2\int \left(\sqrt{t^4-1} - \frac{t^4}{\sqrt{t^4-1}}\right) dt\right] && \text{(O.3)}\\
&= \frac{1}{3}\left[t\sqrt{t^4-1} - 2\int \frac{dt}{\sqrt{t^4-1}}\right]. && \text{(O.4)}
\end{aligned}$$

From Abramowitz and Stegun p.593,597, and Gradshteyn and Rhyzhik p.905,909, the integral in this equation can be written in terms of an elliptic integral

$$\begin{aligned}
\int_1^x \frac{dt}{\sqrt{t^4-1}} &= \frac{1}{\sqrt{2}} F(\varphi,\pi/4) \quad , \quad \cos(\varphi) = \frac{1}{x} && \text{(O.5)}\\
\int_x^1 \frac{dt}{\sqrt{1-t^4}} &= \frac{1}{\sqrt{2}} F(\varphi,\pi/4) \quad , \quad \cos(\varphi) = x && \text{(O.6)}
\end{aligned}$$

where

$$\begin{aligned}
m &= k^2 = \sin^2(\alpha), \quad m_1 = k'^2 = 1-m, && \text{(O.7)}\\
K &= K(m) = F(\pi/2, m), && \text{(O.8)}\\
F(-\phi, m) &= -F(\phi, m) \quad , \quad F(n\pi \pm \phi, m) = 2nK \pm F(\phi, m), && \text{(O.9)}\\
F(0,\alpha) &= 0, && \text{(O.10)}\\
F(\pi/2,\pi/4) &= K(\sin(\pi/4)) = \frac{[\Gamma(1/4)]^2}{4\sqrt{\pi}} = 1.8541, && \text{(O.11)}\\
F(\pi,\pi/4) &= 2K(\sin(\pi/4)) + F(0,\pi/4) = \frac{[\Gamma(1/4)]^2}{2\sqrt{\pi}} = 3.7082. && \text{(O.12)}
\end{aligned}$$

Setting the constant of integration so that $V(r) \rightarrow r^3/3$ as $r \rightarrow \infty$ gives

$$V = \begin{cases} \frac{1}{3}\left[r\sqrt{r^4 - r_0^4} - r_0^3\sqrt{2}\left(F\left(\arccos\left(\frac{r_0}{r}\right),\frac{\pi}{4}\right) - \frac{[\Gamma(1/4)]^2}{4\sqrt{\pi}}\right)\right], r \geq r_0 \\ \frac{i}{3}\left[r\sqrt{r_0^4 - r^4} - r_0^3\sqrt{2}\left(F\left(\arccos\left(\frac{r}{r_0}\right),\frac{\pi}{4}\right) + \frac{i[\Gamma(1/4)]^2}{4\sqrt{\pi}}\right)\right], r \leq r_0. \end{cases} \qquad \text{(O.13)}$$

Using (O.10,O.11) gives

$$V(0) = (1-i)\frac{r_0^3\sqrt{2}\,[\Gamma(1/4)]^2}{12\sqrt{\pi}} = (1-i)0.87402 r_0^3, \quad (O.14)$$

$$V(r_0) = \frac{r_0^3\sqrt{2}\,[\Gamma(1/4)]^2}{12\sqrt{\pi}} = 0.87402 r_0^3, \quad (O.15)$$

$$\hat{V}(r_0) = \frac{\sqrt{2}\,[\Gamma(1/4)]^2}{6\sqrt{\pi}} - \frac{2}{3} = 1.08137. \quad (O.16)$$

These last results can be verified using an infinite power series

$$\sqrt{1-r_0^4/r^4} = 1 - \frac{r_0^4}{2r^4} ... - \frac{(2i)!}{[i!]^2 4^i (2i-1)}\left(\frac{r_0}{r}\right)^{4i}, \quad (O.17)$$

$$\frac{V}{r^3} = \frac{1}{r^3}\int r^2\sqrt{1-r_0^4/r^4}\,dr = \frac{1}{3} + \frac{r_0^4}{2r^4} ... + \frac{(2i)!}{[i!]^2 4^i (2i-1)(4i-3)}\left(\frac{r_0}{r}\right)^{4i}, \quad (O.18)$$

$$\hat{V} = \frac{2r}{r_0^4}\left(V - \frac{r^3}{3}\right) = 1 + \frac{r_0^4}{20r^4} ... + \frac{(2i)!}{i!(i+1)!4^i(4i+1)}\left(\frac{r_0}{r}\right)^{4i}. \quad (O.19)$$

Since surface area vanishes at $r = r_0$ from (3.15), the proper definition of the origin is really $r = r_0$ instead of $r = 0$. So the complex $\hat{V}(r)$ and imaginary $\check{c}$ for $r < r_0$ is not important. As we have shown in §3.1, most of the relevant fields are not singular at our shifted origin. However g_{11} is singular at the origin and $\check{c} = 0$ there. Also, to avoid a curvature singularity we must have the Weyl tensor components be finite everywhere. From §6.1, all of the curvature scalars are zero except for,

$$\Psi_2 = -\frac{1}{\check{c}}\left(1 - \frac{2r_0^4}{r^4}\right)\left(\frac{m}{r^3} + \frac{\Lambda_z V}{2r^3} - \frac{\Lambda_z \check{c}}{6}\right) - \frac{\Lambda_z r_0^4}{6r^4} - \frac{r_0^4}{2\check{c} r^6}. \quad (O.20)$$

Evaluating this at $r = r_0$ gives

$$\Psi_2 = -\frac{1}{\check{c}}\left(1 - \frac{2r_0^4}{r_0^4}\right)\left(\frac{m}{r_0^3} + \frac{0.87402\Lambda_z r_0^3}{2r_0^3}\right) - \frac{\Lambda_z}{6} - \frac{\Lambda_z r_0^4}{6r_0^4} - \frac{r_0^4}{2\check{c} r_0^6} \quad (O.21)$$

$$= \frac{1}{\check{c}}\left(\frac{m}{r_0^3} + \frac{0.87402\Lambda_z}{2} - \frac{1}{2r_0^2}\right) - \frac{\Lambda_z}{3}. \quad (O.22)$$

Since $\check{c} = 0$ at $r = r_0$, the only way to avoid a singularity in Ψ_2 is if

$$m = \frac{r_0}{2}(1 - 0.87402\Lambda_z r_0^2). \quad (O.23)$$

From (2.36,2.12), for an elementary charge we would have $m \sim r_0 \sim 10^{-33} cm$, whereas the mass of an electron in geometrical units is $M_e = Gm_e/c^2 = 7\times 10^{-56} cm$. Since Λ_z is negative when Q is real, it is not possible to get the two terms in the parenthesis to partially cancel one another. So it does not appear that a singularity in the Weyl scalars can be avoided by assuming the mass and charge of an electron. It is possible that the physical charge and mass of an electron are actually renormalized values, and that things would work out if one used "bare" charges and masses instead. However, even if this happened to work out, it is not clear that we can avoid a singularity in the Weyl tensor itself, since the tetrads are actually singular at the origin.

Now let us redo the calculation with $r_e^4 = -r_0^4$, which corresponds from (3.16) to the supersymmetry case with $\Lambda_b < 0$. Changing variables with $t = r/r_e$ gives

$$\hat{V} = -\frac{2r}{r_e^4}\left(V - \frac{r^3}{3}\right), \qquad V = \int \sqrt{r^4 + r_e^4}\, dr = r_e^3 \int \sqrt{t^4+1}\, dt. \quad \text{(O.24)}$$

Integration by parts gives

$$\begin{aligned} \int \sqrt{t^4+1}\, dt &= \frac{2}{3}\int \sqrt{t^4+1}\, dt + \frac{1}{3}\left[t\sqrt{t^4+1} - \int \frac{2t^4 dt}{\sqrt{t^4+1}}\right] && \text{(O.25)} \\ &= \frac{1}{3}\left[t\sqrt{t^4+1} + 2\int \left(\sqrt{t^4+1} - \frac{t^4}{\sqrt{t^4+1}}\right) dt\right] && \text{(O.26)} \\ &= \frac{1}{3}\left[t\sqrt{t^4+1} + 2\int \frac{dt}{\sqrt{t^4+1}}\right]. && \text{(O.27)} \end{aligned}$$

From Abramowitz and Stegun p.593,597, and Gradshteyn and Rhyzhik p.905,909, the integral in this equation can be written in terms of an elliptic integral

$$-\int_x^\infty \frac{dt}{\sqrt{t^4+1}} = -\frac{1}{2}F(\varphi, \pi/4) \quad , \quad \cos(\varphi) = \frac{x^2-1}{x^2+1}. \quad \text{(O.28)}$$

Using a trigonometric identity we have

$$\cos(\varphi) = \frac{\cot^2(\varphi/2) - 1}{\cot^2(\varphi/2) + 1} \quad \Rightarrow \quad x = \cot(\varphi/2) \quad , \quad \varphi = 2\,\mathrm{arccot}(x). \quad \text{(O.29)}$$

Setting the constant of integration so that $V(r) \to r^3/3$ as $r \to \infty$ gives

$$V = \frac{1}{3}\left[r\sqrt{r^4 + r_e^4} - r_e^3 F\left(2\arctan\left(\frac{r_e}{r}\right), \frac{\pi}{4}\right)\right]. \tag{O.30}$$

Using (O.10,O.11) gives

$$V(0) = -\frac{r_e^3\,[\Gamma(1/4)]^2}{6\sqrt{\pi}}, \tag{O.31}$$

$$V(r_e) = \frac{r_e^3}{3}\left[\sqrt{2} - \frac{[\Gamma(1/4)]^2}{4\sqrt{\pi}}\right] = -.14662 r_e^3, \tag{O.32}$$

$$\hat{V}(r_e) = \frac{2}{3}\left[1 - \sqrt{2} + \frac{[\Gamma(1/4)]^2}{4\sqrt{\pi}}\right] = .95991. \tag{O.33}$$

These last results can be verified using an infinite power series

$$\sqrt{1 + r_e^4/r^4} = 1 + \frac{r_e^4}{2r^4} \ldots - \frac{(-1)^i (2i)!}{[i!]^2 4^i (2i-1)} \left(\frac{r_e}{r}\right)^{4i}, \tag{O.34}$$

$$\frac{V}{r^3} = \frac{1}{r^3}\int r^2 \sqrt{1 + r_e^4/r^4}\; dr = \frac{1}{3} - \frac{r_e^4}{2r^4} \ldots + \frac{(-1)^i (2i)!}{[i!]^2 4^i (2i-1)(4i-3)} \left(\frac{r_e}{r}\right)^{4i}, \tag{O.35}$$

$$\hat{V} = \frac{2r}{r_e^4}\left(\frac{r^3}{3} - V\right) = 1 - \frac{r_e^4}{20r^4} \ldots + \frac{(-1)^i (2i)!}{i!(i+1)! 4^i (4i+1)} \left(\frac{r_e}{r}\right)^{4i}. \tag{O.36}$$

Appendix P

The electric monopole solution in alternative coordinates

Here we investigate the charged solution (3.1-3.7), but with a different radial coordinate where the origin is at $\rho = 0$,

$$\rho = r\sqrt{\check{c}} = (r^4 - 2Q^2/\Lambda_b)^{1/4}, \qquad r = (\rho^4 + 2Q^2/\Lambda_b)^{1/4}, \tag{P.1}$$

$$\check{c} = \sqrt{1 - \frac{2Q^2}{\Lambda_b(\rho^4 + 2Q^2/\Lambda_b)}} = \frac{\rho^2}{\sqrt{\rho^4 + 2Q^2/\Lambda_b}}, \tag{P.2}$$

$$\frac{d\rho}{dr} = \frac{r^3}{(r^4 - 2Q^2/\Lambda_b)^{3/4}} = \frac{1}{\check{c}^{3/2}}, \qquad \frac{dr}{d\rho} = \check{c}^{3/2}, \qquad \frac{d^2r}{d\rho^2} = \frac{6Q^2\check{c}^{7/2}}{\Lambda_b\rho^5}. \tag{P.3}$$

With this new radial coordinate the solution becomes

$$ds^2 = \check{c}adt^2 - \frac{\check{c}^2}{a}d\rho^2 - \rho^2 d\theta^2 - \rho^2 sin^2\theta d\phi^2, \tag{P.4}$$

$$f^{10} = \frac{Q}{\check{c}^{3/2}\rho^2}, \qquad \sqrt{-N} = \check{c}^2\rho^2 sin\,\theta, \qquad \sqrt{-g} = \check{c}^3\rho^2 sin\,\theta, \tag{P.5}$$

$$F_{01} = -A_0' = \frac{Q\check{c}^{5/2}}{\rho^2}\left[1 + \frac{4M\check{c}^{3/2}}{\Lambda_b\rho^3} - \frac{4\Lambda}{3\Lambda_b} + 2\left(\check{c} - 1 - \frac{Q^2\hat{V}\check{c}^2}{\Lambda_b\rho^4}\right)\left(1 - \frac{\Lambda}{\Lambda_b}\right)\right], \tag{P.6}$$

$$a = 1 - \frac{2M\check{c}^{1/2}}{\rho} - \frac{\Lambda\rho^2}{3\check{c}} + \frac{Q^2\hat{V}\check{c}}{\rho^2}\left(1 - \frac{\Lambda}{\Lambda_b}\right), \tag{P.7}$$

where ($'$) means $\partial/\partial\rho$, and $\hat{V}$ is very close to one for ordinary radii,

$$\hat{V} \;=\; \frac{\rho\Lambda_b}{Q^2\check{c}^{1/2}}\left(\int\rho^2\check{c}^{3/2}\,d\rho-\frac{\rho^3}{3\check{c}^{3/2}}\right), \tag{P.8}$$

and the nonzero connections are

$$\begin{aligned}
&\tilde{\Gamma}^1_{00}=\frac{aa'}{\check{c}}-\frac{4a^2Q^2\check{c}}{\Lambda_b\,\rho^5}\;\;,\;\tilde{\Gamma}^0_{10}=\tilde{\Gamma}^0_{01}=\frac{a'}{2a}\;\;,\;\tilde{\Gamma}^1_{11}=\frac{-a'}{2a}+\frac{6Q^2\check{c}^2}{\Lambda_b\rho^5},\\
&\tilde{\Gamma}^2_{12}=\tilde{\Gamma}^2_{21}=\tilde{\Gamma}^3_{13}=\tilde{\Gamma}^3_{31}=\frac{\check{c}^2}{\rho},\\
&\tilde{\Gamma}^1_{22}=-\frac{a\rho}{\check{c}^2}\;\;,\;\tilde{\Gamma}^1_{33}=-\frac{a\rho\sin^2\theta}{\check{c}^2}\;\;,\;\tilde{\Gamma}^3_{23}=\tilde{\Gamma}^3_{32}=\cot\theta\;\;,\;\tilde{\Gamma}^2_{33}=-\sin\theta\cos\theta,\\
&\tilde{\Gamma}^2_{02}=-\tilde{\Gamma}^2_{20}=\tilde{\Gamma}^3_{03}=-\tilde{\Gamma}^3_{30}=-\frac{a\sqrt{2}\,iQ\check{c}^{3/2}}{\sqrt{\Lambda_b}\,\rho^3}\;\;,\;\tilde{\Gamma}^1_{10}=-\tilde{\Gamma}^1_{01}=-\frac{2a\sqrt{2}\,iQ\check{c}^{3/2}}{\sqrt{\Lambda_b}\,\rho^3}.
\end{aligned} \tag{P.9}$$

For this radial coordinate, $g_{\mu\nu}$ has a finite value and derivative at the origin, although $g^{\mu\nu}$ does not. Also, the fields $N_{\mu\nu}$, $N^{\dashv\nu\mu}$, $\sqrt{-N}$, $\sqrt{-g}$, A_ν, $\sqrt{-g}f^{\nu\mu}$, $\sqrt{-g}f_{\nu\mu}$, $\sqrt{-g}g^{\nu\mu}$, $\sqrt{-g}g_{\nu\mu}$, and the functions "a" and $\hat{V}$ all have finite values and derivatives at the origin, because as before $\hat{V}(0)=\sqrt{2}\,[\Gamma(1/4)]^2/6\sqrt{\pi}-2/3 = 1.08137$. The fields $F_{\nu\mu}$ and $\sqrt{-g}\,\tilde{\mathcal{R}}_{\nu\mu}$ are also finite at the origin, although $\tilde{\Gamma}^\alpha_{\mu\nu}$ is not.

Appendix Q

The electromagnetic plane-wave solution in alternative coordinates

Here we consider the plane-wave solution in §3.2 for a couple different coordinate systems. First, to make the solution look more familiar we will transform it to ordinary x, y, z, t coordinates. With the conversion $z = (v - u)/\sqrt{2}, \quad t = (v + u)/\sqrt{2}$ we have

$$T^{\sigma}{}_{\mu} = \frac{\partial \breve{X}^{\sigma}}{\partial X^{\mu}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \tag{Q.1}$$

$$T^{-1\sigma}{}_{\mu} = \frac{\partial X^{\sigma}}{\partial \breve{X}^{\mu}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{Q.2}$$

Then from (3.17,3.18,3.19) we get,

$$g_{\alpha\sigma}T^{-1\sigma}{}_{\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -H/\sqrt{2}+1/\sqrt{2} & H/\sqrt{2}+1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \tag{Q.3}$$

$$\breve{g}_{\nu\mu} = T_{\nu}^{-1\alpha} g_{\alpha\sigma}T^{-1\sigma}{}_{\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1+H/2 & -H/2 \\ 0 & 0 & -H/2 & 1+H/2 \end{pmatrix}, \tag{Q.4}$$

$$\sqrt{-\breve{g}} = \sqrt{-\breve{N}} = 1, \tag{Q.5}$$

$$f^{\alpha\sigma}T_{\sigma}{}^{\mu} = \left(\frac{1}{1}\right) \begin{pmatrix} 0 & 0 & \breve{f}_x & \breve{f}_x \\ 0 & 0 & \breve{f}_y & \breve{f}_y \\ 0 & 0 & 0 & 0 \\ -\sqrt{2}\breve{f}_x & -\sqrt{2}\breve{f}_y & 0 & 0 \end{pmatrix}, \tag{Q.6}$$

$$\sqrt{-\breve{g}}\breve{f}^{\nu\mu} = (1)T^{\nu}{}_{\alpha}f^{\alpha\sigma}T_{\sigma}{}^{\mu} = \begin{pmatrix} 0 & 0 & \breve{f}_x & \breve{f}_x \\ 0 & 0 & \breve{f}_y & \breve{f}_y \\ -\breve{f}_x & -\breve{f}_y & 0 & 0 \\ -\breve{f}_x & -\breve{f}_y & 0 & 0 \end{pmatrix}, \tag{Q.7}$$

$$f_{\alpha\sigma}T^{-1\sigma}{}_{\mu} = \begin{pmatrix} 0 & 0 & \breve{f}_x & -\breve{f}_x \\ 0 & 0 & \breve{f}_y & -\breve{f}_y \\ \sqrt{2}\breve{f}_x & \sqrt{2}\breve{f}_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{Q.8}$$

$$\breve{f}_{\nu\mu} = T_{\nu}^{-1\alpha} f_{\alpha\sigma} T^{-1\sigma}{}_{\mu} = \begin{pmatrix} 0 & 0 & \breve{f}_x & -\breve{f}_x \\ 0 & 0 & \breve{f}_y & -\breve{f}_y \\ -\breve{f}_x & -\breve{f}_y & 0 & 0 \\ \breve{f}_x & \breve{f}_y & 0 & 0 \end{pmatrix}, \tag{Q.9}$$

$$\breve{A}_{\mu} = A_{\sigma} T^{-1\sigma}{}_{\mu} = (0, 0, x\breve{f}_x + y\breve{f}_y, -x\breve{f}_x - y\breve{f}_y). \tag{Q.10}$$

From page 61 of [75] we have

$$f_{\mu\nu} = \begin{pmatrix} 0 & -B_z & B_y & -E_x \\ B_x & 0 & -B_x & -E_y \\ -B_y & B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{pmatrix}, \tag{Q.11}$$

so the electric and magnetic fields are

$$E = (\breve{f}_x, \breve{f}_y, 0), \qquad B = (-\breve{f}_y, \breve{f}_x, 0). \tag{Q.12}$$

Now let us consider the solution in another type of x, y, u, v coordinates such that it matches the solution on page 961 of [66]. Let us assume some as yet unspecified function $L(u)$, and we will use the notation $L' = \partial L/\partial u$. With the conversion $\breve{x} =$

$x/L,\ \breve{y} = y/L,\ \breve{u} = \sqrt{2}\,u,\ \breve{v} = \sqrt{2}\,v - (x^2+y^2)L'/\sqrt{2}\,L$ we have

$$T^{\sigma}{}_{\mu} \;=\; \frac{\partial \breve{X}^{\sigma}}{\partial X^{\mu}} = \begin{pmatrix} 1/L & 0 & -xL'/L^2 & 0 \\ 0 & 1/L & -yL'/L^2 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -xL'\sqrt{2}/L & -yL'\sqrt{2}/L & (x^2{+}y^2)(L'^2/L^2{-}L''/L)/\sqrt{2} & \sqrt{2} \end{pmatrix}, \tag{Q.13}$$

$$T^{-1\sigma}{}_{\mu} \;=\; \frac{\partial X^{\sigma}}{\partial \breve{X}^{\mu}} = \begin{pmatrix} L & 0 & xL'/\sqrt{2}L & 0 \\ 0 & L & yL'/\sqrt{2}L & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ xL' & yL' & (x^2{+}y^2)(L'^2/L^2{+}L''/L)/2\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{Q.14}$$

Then from (3.17,3.18,3.19) we get,

$$g_{\alpha\sigma}T^{-1\sigma}{}_{\mu} = \begin{pmatrix} -L & 0 & -xL'/\sqrt{2}L & 0 \\ 0 & -L & -yL'/\sqrt{2}L & 0 \\ xL' & yL' & H/\sqrt{2} + (x^2{+}y^2)(L'^2/L^2{+}L''/L)/2\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 \end{pmatrix}, \tag{Q.15}$$

$$\breve{g}_{\nu\mu}{=}T^{-1\alpha}_{\nu}g_{\alpha\sigma}T^{-1\sigma}{}_{\mu} = \begin{pmatrix} -L^2 & 0 & 0 & 0 \\ 0 & -L^2 & 0 & 0 \\ 0 & 0 & (x^2{+}y^2)Z + h_+x^2 + h_\times xy - h_+y^2 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}, \tag{Q.16}$$

$$\text{where } Z = \breve{f}_x^2 + \breve{f}_y^2 + L''/2L, \tag{Q.17}$$

$$\sqrt{-\breve{g}} = \sqrt{-\breve{N}} = L^2/2, \tag{Q.18}$$

$$f^{\alpha\sigma}T_{\sigma}{}^{\mu} = \left(\frac{1}{1}\right)\begin{pmatrix} 0 & 0 & 0 & 2\check{f}_x \\ 0 & 0 & 0 & 2\check{f}_y \\ 0 & 0 & 0 & 0 \\ -\sqrt{2}\check{f}_x/L & -\sqrt{2}\check{f}_y/L & 0 & 2(x\check{f}_x+y\check{f}_y)L'/L \end{pmatrix}, \tag{Q.19}$$

$$\sqrt{-\breve{g}}\check{f}^{\nu\mu} = (L^2/2)T^{\nu}{}_{\alpha}f^{\alpha\sigma}T_{\sigma}{}^{\mu} = \begin{pmatrix} 0 & 0 & 0 & L\check{f}_x \\ 0 & 0 & 0 & L\check{f}_y \\ 0 & 0 & 0 & 0 \\ -L\check{f}_x & -L\check{f}_y & 0 & 0 \end{pmatrix}, \tag{Q.20}$$

$$f_{\alpha\sigma}T^{-1\sigma}{}_{\mu} = \begin{pmatrix} 0 & 0 & -\check{f}_x & 0 \\ 0 & 0 & -\check{f}_y & 0 \\ \sqrt{2}\check{f}_xL & \sqrt{2}\check{f}_yL & (x\check{f}_x+y\check{f}_y)L'/L & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{Q.21}$$

$$\check{f}_{\nu\mu}=T_{\nu}^{-1\alpha}f_{\alpha\sigma}T^{-1\sigma}{}_{\mu} = \begin{pmatrix} 0 & 0 & -L\check{f}_x & 0 \\ 0 & 0 & -L\check{f}_y & 0 \\ L\check{f}_x & L\check{f}_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{Q.22}$$

$$\check{A}_{\mu} = A_{\sigma}T^{-1\sigma}{}_{\mu} = (0,0,-x\check{f}_x - y\check{f}_y, 0). \tag{Q.23}$$

Let us choose $L(u)$ in (Q.17) such that $Z=0$. Equation (Q.17) with $Z=0$ is a 2nd order differential equation, and ignoring boundary condition issues, this equation can be solved for arbitrary functions $\check{f}_x(u), \check{f}_y(u)$. Then for the special case $h_{\times}=h_{+}=0$ where there is no gravitational wave component, we get the solution in [66].

Appendix R

Some properties of the non-symmetric Ricci tensor

Substituting $\tilde{\Gamma}^\alpha_{\nu\mu}=\Gamma^\alpha_{\nu\mu}+\Upsilon^\alpha_{\nu\mu}$ from (2.61,2.20) into (7.21) gives

$$\begin{aligned}
\mathcal{R}_{\nu\mu}(\tilde{\Gamma}) &= \tilde{\Gamma}^\alpha_{\nu\mu,\alpha} - \tilde{\Gamma}^\alpha_{\alpha(\nu,\mu)} + \frac{1}{2}\tilde{\Gamma}^\sigma_{\nu\mu}\tilde{\Gamma}^\alpha_{\sigma\alpha} + \frac{1}{2}\tilde{\Gamma}^\alpha_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\nu\mu} - \tilde{\Gamma}^\sigma_{\nu\alpha}\tilde{\Gamma}^\alpha_{\sigma\mu} && \text{(R.1)}\\
&= (\Gamma^\alpha_{\nu\mu,\alpha} + \Upsilon^\alpha_{\nu\mu,\alpha}) - (\Gamma^\alpha_{\alpha(\nu,\mu)} + \Upsilon^\alpha_{\alpha(\nu,\mu)}) - (\Gamma^\sigma_{\nu\alpha} + \Upsilon^\sigma_{\nu\alpha})(\Gamma^\alpha_{\sigma\mu} + \Upsilon^\alpha_{\sigma\mu})\\
&\quad + \frac{1}{2}(\Gamma^\sigma_{\nu\mu} + \Upsilon^\sigma_{\nu\mu})(\Gamma^\alpha_{\sigma\alpha} + \Upsilon^\alpha_{\sigma\alpha}) + \frac{1}{2}(\Gamma^\alpha_{\sigma\alpha} + \Upsilon^\alpha_{\sigma\alpha})(\Gamma^\sigma_{\nu\mu} + \Upsilon^\sigma_{\nu\mu}) && \text{(R.2)}\\
&= R_{\nu\mu}(\Gamma) + \Upsilon^\alpha_{\nu\mu,\alpha} - \Upsilon^\alpha_{\alpha(\nu,\mu)} - \Gamma^\sigma_{\nu\alpha}\Upsilon^\alpha_{\sigma\mu} - \Upsilon^\sigma_{\nu\alpha}\Gamma^\alpha_{\sigma\mu} - \Upsilon^\sigma_{\nu\alpha}\Upsilon^\alpha_{\sigma\mu}\\
&\quad + \frac{1}{2}(\Gamma^\sigma_{\nu\mu}\Upsilon^\alpha_{\sigma\alpha} + \Upsilon^\sigma_{\nu\mu}\Gamma^\alpha_{\sigma\alpha} + \Upsilon^\sigma_{\nu\mu}\Upsilon^\alpha_{\sigma\alpha} + \Gamma^\alpha_{\sigma\alpha}\Upsilon^\sigma_{\nu\mu} + \Upsilon^\alpha_{\sigma\alpha}\Gamma^\sigma_{\nu\mu} + \Upsilon^\alpha_{\sigma\alpha}\Upsilon^\sigma_{\nu\mu}) && \text{(R.3)}\\
&= R_{\nu\mu}(\Gamma) + \Upsilon^\alpha_{\nu\mu;\alpha} - \Upsilon^\alpha_{\alpha(\nu;\mu)} - \Upsilon^\sigma_{\nu\alpha}\Upsilon^\alpha_{\sigma\mu} + \frac{1}{2}\Upsilon^\sigma_{\nu\mu}\Upsilon^\alpha_{\sigma\alpha} + \frac{1}{2}\Upsilon^\alpha_{\sigma\alpha}\Upsilon^\sigma_{\nu\mu}. && \text{(R.4)}
\end{aligned}$$

For the Abelian case this gives (2.65,2.66),

$$\begin{aligned}
\mathcal{R}_{\nu\mu}(\tilde{\Gamma}) &= R_{\nu\mu}(\Gamma) + \Upsilon^\alpha_{\nu\mu;\alpha} - \Upsilon^\alpha_{\alpha(\nu;\mu)} - \Upsilon^\sigma_{\nu\alpha}\Upsilon^\alpha_{\sigma\mu} + \Upsilon^\sigma_{\nu\mu}\Upsilon^\alpha_{\sigma\alpha}, && \text{(R.5)}\\
\mathcal{R}_{(\nu\mu)}(\tilde{\Gamma}) &= R_{\nu\mu}(\Gamma) + \Upsilon^\alpha_{(\nu\mu);\alpha} - \Upsilon^\alpha_{\alpha(\nu;\mu)} - \Upsilon^\sigma_{(\nu\alpha)}\Upsilon^\alpha_{(\sigma\mu)} - \Upsilon^\sigma_{[\nu\alpha]}\Upsilon^\alpha_{[\sigma\mu]} + \Upsilon^\sigma_{(\nu\mu)}\Upsilon^\alpha_{\sigma\alpha}, && \text{(R.6)}\\
\mathcal{R}_{[\nu\mu]}(\tilde{\Gamma}) &= \Upsilon^\alpha_{[\nu\mu];\alpha} - \Upsilon^\sigma_{(\nu\alpha)}\Upsilon^\alpha_{[\sigma\mu]} - \Upsilon^\sigma_{[\nu\alpha]}\Upsilon^\alpha_{(\sigma\mu)} + \Upsilon^\sigma_{[\nu\mu]}\Upsilon^\alpha_{\sigma\alpha}. && \text{(R.7)}
\end{aligned}$$

Substituting the $SU(2)$ gauge transformation $\widehat{\Gamma}^{\alpha}_{\nu\mu} \to U\widehat{\Gamma}^{\alpha}_{\nu\mu}U^{-1} + 2\delta^{\alpha}_{[\nu}U_{,\mu]}U^{-1}$ from (7.29) into $\mathcal{R}_{\nu\mu}$ proves the result (7.41),

$$
\begin{aligned}
\mathcal{R}_{\nu\mu}(\widehat{\Gamma}') &= \widehat{\Gamma}^{\alpha}_{\nu\mu,\alpha} - \widehat{\Gamma}^{\alpha}_{(\alpha(\nu),\mu)} + \frac{1}{2}\widehat{\Gamma}^{\sigma}_{\nu\mu}\widehat{\Gamma}^{\alpha}_{(\sigma\alpha)} + \frac{1}{2}\widehat{\Gamma}^{\alpha}_{(\sigma\alpha)}\widehat{\Gamma}^{\sigma}_{\nu\mu} - \widehat{\Gamma}^{\sigma}_{\nu\alpha}\widehat{\Gamma}^{\alpha}_{\sigma\mu} - \frac{\widehat{\Gamma}^{\tau}_{[\tau\nu]}\widehat{\Gamma}^{\rho}_{[\rho\mu]}}{(n-1)} \qquad \text{(R.8)}\\
&= \Big(U\widehat{\Gamma}^{\alpha}_{\nu\mu}U^{-1} + \delta^{\alpha}_{\nu}U_{,\mu}U^{-1} - \delta^{\alpha}_{\mu}U_{,\nu}U^{-1}\Big)_{,\alpha}\\
&- \frac{1}{2}(U\widehat{\Gamma}^{\alpha}_{(\alpha\nu)}U^{-1})_{,\mu} - \frac{1}{2}(U\widehat{\Gamma}^{\alpha}_{(\alpha\mu)}U^{-1})_{,\nu}\\
&+ \frac{1}{2}\Big(U\widehat{\Gamma}^{\sigma}_{\nu\mu}U^{-1} + \delta^{\sigma}_{\nu}U_{,\mu}U^{-1} - \delta^{\sigma}_{\mu}U_{,\nu}U^{-1}\Big)U\widehat{\Gamma}^{\alpha}_{(\sigma\alpha)}U^{-1}\\
&+ \frac{1}{2}U\widehat{\Gamma}^{\alpha}_{(\sigma\alpha)}U^{-1}\Big(U\widehat{\Gamma}^{\sigma}_{\nu\mu}U^{-1} + \delta^{\sigma}_{\nu}U_{,\mu}U^{-1} - \delta^{\sigma}_{\mu}U_{,\nu}U^{-1}\Big)\\
&- \Big(U\widehat{\Gamma}^{\sigma}_{\nu\alpha}U^{-1} + \delta^{\sigma}_{\nu}U_{,\alpha}U^{-1} - \delta^{\sigma}_{\alpha}U_{,\nu}U^{-1}\Big)\Big(U\widehat{\Gamma}^{\alpha}_{\sigma\mu}U^{-1} + \delta^{\alpha}_{\sigma}U_{,\mu}U^{-1} - \delta^{\alpha}_{\mu}U_{,\sigma}U^{-1}\Big)\\
&- \frac{1}{(n-1)}\Big(U\widehat{\Gamma}^{\tau}_{[\tau\nu]}U^{-1} + (n-1)\,U_{,\nu}U^{-1}\Big)\Big(U\widehat{\Gamma}^{\rho}_{[\rho\mu]}U^{-1} + (n-1)\,U_{,\mu}U^{-1}\Big) \qquad \text{(R.9)}\\
&= U\left(\widehat{\Gamma}^{\alpha}_{\nu\mu,\alpha} - \widehat{\Gamma}^{\alpha}_{(\alpha(\nu),\mu)} + \frac{1}{2}\widehat{\Gamma}^{\sigma}_{\nu\mu}\widehat{\Gamma}^{\alpha}_{(\sigma\alpha)} + \frac{1}{2}\widehat{\Gamma}^{\alpha}_{(\sigma\alpha)}\widehat{\Gamma}^{\sigma}_{\nu\mu} - \widehat{\Gamma}^{\sigma}_{\nu\alpha}\widehat{\Gamma}^{\alpha}_{\sigma\mu} - \frac{\widehat{\Gamma}^{\tau}_{[\tau\nu]}\widehat{\Gamma}^{\rho}_{[\rho\mu]}}{(n-1)}\right)U^{-1}\\
&+ U_{,\alpha}\widehat{\Gamma}^{\alpha}_{\nu\mu}U^{-1} + U\widehat{\Gamma}^{\alpha}_{\nu\mu}U^{-1}_{,\alpha} + U_{,\mu}U^{-1}_{,\nu} - U_{,\nu}U^{-1}_{,\mu}\\
&- \frac{1}{2}U_{,\mu}\widehat{\Gamma}^{\alpha}_{(\alpha\nu)}U^{-1} - \frac{1}{2}U\widehat{\Gamma}^{\alpha}_{(\alpha\nu)}U^{-1}_{,\mu} - \frac{1}{2}U_{,\nu}\widehat{\Gamma}^{\alpha}_{(\alpha\mu)}U^{-1} - \frac{1}{2}U\widehat{\Gamma}^{\alpha}_{(\alpha\mu)}U^{-1}_{,\nu}\\
&+ \frac{1}{2}U_{,\mu}\widehat{\Gamma}^{\alpha}_{(\nu\alpha)}U^{-1} - \frac{1}{2}U_{,\nu}\widehat{\Gamma}^{\alpha}_{(\mu\alpha)}U^{-1}\\
&- \frac{1}{2}U\widehat{\Gamma}^{\alpha}_{(\nu\alpha)}U^{-1}_{,\mu} + \frac{1}{2}U\widehat{\Gamma}^{\alpha}_{(\mu\alpha)}U^{-1}_{,\nu}\\
&+ U\widehat{\Gamma}^{\sigma}_{\nu\sigma}U^{-1}_{,\mu} - U\widehat{\Gamma}^{\sigma}_{\nu\mu}U^{-1}_{,\sigma} - U_{,\alpha}\widehat{\Gamma}^{\alpha}_{\nu\mu}U^{-1} + U_{,\nu}\widehat{\Gamma}^{\alpha}_{\alpha\mu}U^{-1} + (2-n)U_{,\nu}U^{-1}_{,\mu} - U_{,\mu}U^{-1}_{,\nu}\\
&+ U\widehat{\Gamma}^{\tau}_{[\tau\nu]}U^{-1}_{,\mu} - U_{,\nu}\widehat{\Gamma}^{\rho}_{[\rho\mu]}U^{-1} + (n-1)U_{,\nu}U^{-1}_{,\mu} \qquad \text{(R.10)}\\
&= U\mathcal{R}_{\nu\mu}(\widehat{\Gamma})U^{-1}. \qquad \text{(R.11)}
\end{aligned}
$$

For the special case $U = Ie^{-i\varphi}$ we also get the property (2.17) that $\mathcal{R}_{\nu\mu}$ is invariant under a $U(1)$ gauge transformation $\widehat{\Gamma}^{\alpha}_{\rho\tau} \to \widehat{\Gamma}^{\alpha}_{\rho\tau} + \delta^{\alpha}_{[\rho}\varphi_{,\tau]}$.

Substituting $\widehat{\Gamma}^\alpha_{\nu\mu}=\tilde{\Gamma}^\alpha_{\nu\mu}+(\delta^\alpha_\mu\mathcal{A}_\nu-\delta^\alpha_\nu\mathcal{A}_\mu)\sqrt{-2\Lambda_b}$ from (7.16) into $\mathcal{R}_{\nu\mu}$ using $\tilde{\Gamma}^\alpha_{\nu\alpha}=\widehat{\Gamma}^\alpha_{(\nu\alpha)}=\tilde{\Gamma}^\alpha_{\alpha\nu}$ from (7.18) and the notation $[A,B]=AB-BA$ gives (7.19),

$$
\begin{aligned}
\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) &= \widehat{\Gamma}^\alpha_{\nu\mu,\alpha}-\widehat{\Gamma}^\alpha_{(\alpha(\nu),\mu)}+\frac{1}{2}\widehat{\Gamma}^\sigma_{\nu\mu}\widehat{\Gamma}^\alpha_{(\sigma\alpha)}+\frac{1}{2}\widehat{\Gamma}^\alpha_{(\sigma\alpha)}\widehat{\Gamma}^\sigma_{\nu\mu}-\widehat{\Gamma}^\sigma_{\nu\alpha}\widehat{\Gamma}^\alpha_{\sigma\mu}-\frac{\widehat{\Gamma}^\tau_{[\tau\nu]}\widehat{\Gamma}^\rho_{[\rho\mu]}}{(n-1)} \quad \text{(R.12)}\\
&= \left(\tilde{\Gamma}^\alpha_{\nu\mu}+(\delta^\alpha_\mu\mathcal{A}_\nu-\delta^\alpha_\nu\mathcal{A}_\mu)\sqrt{-2\Lambda_b}\right)_{,\alpha}-\tilde{\Gamma}^\alpha_{(\alpha(\nu),\mu)}\\
&\quad+\frac{1}{2}\left(\tilde{\Gamma}^\sigma_{\nu\mu}+(\delta^\sigma_\mu\mathcal{A}_\nu-\delta^\sigma_\nu\mathcal{A}_\mu)\sqrt{-2\Lambda_b}\right)\tilde{\Gamma}^\alpha_{(\sigma\alpha)}\\
&\quad+\frac{1}{2}\tilde{\Gamma}^\alpha_{(\sigma\alpha)}\left(\tilde{\Gamma}^\sigma_{\nu\mu}+(\delta^\sigma_\mu\mathcal{A}_\nu-\delta^\sigma_\nu\mathcal{A}_\mu)\sqrt{-2\Lambda_b}\right)\\
&\quad-\left(\tilde{\Gamma}^\sigma_{\nu\alpha}+(\delta^\sigma_\alpha\mathcal{A}_\nu-\delta^\sigma_\nu\mathcal{A}_\alpha)\sqrt{-2\Lambda_b}\right)\left(\tilde{\Gamma}^\alpha_{\sigma\mu}+(\delta^\alpha_\mu\mathcal{A}_\sigma-\delta^\alpha_\sigma\mathcal{A}_\mu)\sqrt{-2\Lambda_b}\right)\\
&\quad+2(n-1)\Lambda_b\mathcal{A}_\nu\mathcal{A}_\mu \quad \text{(R.13)}\\
&= \tilde{\Gamma}^\alpha_{\nu\mu,\alpha}-\tilde{\Gamma}^\alpha_{\alpha(\nu,\mu)}+\frac{1}{2}\tilde{\Gamma}^\sigma_{\nu\mu}\tilde{\Gamma}^\alpha_{\sigma\alpha}+\frac{1}{2}\tilde{\Gamma}^\alpha_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\nu\mu}-\tilde{\Gamma}^\sigma_{\nu\alpha}\tilde{\Gamma}^\alpha_{\sigma\mu}\\
&\quad+2\mathcal{A}_{[\nu,\mu]}\sqrt{-2\Lambda_b}\\
&\quad+\frac{1}{2}(\delta^\sigma_\mu\mathcal{A}_\nu-\delta^\sigma_\nu\mathcal{A}_\mu)\tilde{\Gamma}^\alpha_{\sigma\alpha}\sqrt{-2\Lambda_b}\\
&\quad+\frac{1}{2}\tilde{\Gamma}^\alpha_{\sigma\alpha}(\delta^\sigma_\mu\mathcal{A}_\nu-\delta^\sigma_\nu\mathcal{A}_\mu)\sqrt{-2\Lambda_b}\\
&\quad-\tilde{\Gamma}^\sigma_{\nu\alpha}(\delta^\alpha_\mu\mathcal{A}_\sigma-\delta^\alpha_\sigma\mathcal{A}_\mu)\sqrt{-2\Lambda_b}\\
&\quad-(\delta^\sigma_\alpha\mathcal{A}_\nu-\delta^\sigma_\nu\mathcal{A}_\alpha)\tilde{\Gamma}^\alpha_{\sigma\mu}\sqrt{-2\Lambda_b}\\
&\quad+2(n-1)\Lambda_b\mathcal{A}_\nu\mathcal{A}_\mu+2\Lambda_b((2-n)\mathcal{A}_\nu\mathcal{A}_\mu-\mathcal{A}_\mu\mathcal{A}_\nu) \quad \text{(R.14)}\\
&= \mathcal{R}_{\nu\mu}(\tilde{\Gamma})+2\mathcal{A}_{[\nu,\mu]}\sqrt{-2\Lambda_b}+2\Lambda_b(\mathcal{A}_\nu\mathcal{A}_\mu-\mathcal{A}_\mu\mathcal{A}_\nu)\\
&\quad+\frac{1}{2}(\tilde{\Gamma}^\alpha_{\mu\alpha}\mathcal{A}_\nu-\mathcal{A}_\nu\tilde{\Gamma}^\alpha_{\mu\alpha})\sqrt{-2\Lambda_b}\\
&\quad+\frac{1}{2}(\tilde{\Gamma}^\alpha_{\nu\alpha}\mathcal{A}_\mu-\mathcal{A}_\mu\tilde{\Gamma}^\alpha_{\nu\alpha})\sqrt{-2\Lambda_b}\\
&\quad+(\mathcal{A}_\alpha\tilde{\Gamma}^\alpha_{\nu\mu}-\tilde{\Gamma}^\sigma_{\nu\mu}\mathcal{A}_\sigma)\sqrt{-2\Lambda_b} \quad \text{(R.15)}\\
&= \mathcal{R}_{\nu\mu}(\tilde{\Gamma})+2\mathcal{A}_{[\nu,\mu]}\sqrt{-2\Lambda_b}+2\Lambda_b[\mathcal{A}_\nu,\mathcal{A}_\mu]\\
&\quad+([\mathcal{A}_\alpha,\tilde{\Gamma}^\alpha_{\nu\mu}]-[\mathcal{A}_{(\nu},\tilde{\Gamma}^\alpha_{\mu)\alpha}])\sqrt{-2\Lambda_b}\,. \quad \text{(R.16)}
\end{aligned}
$$

For the Abelian case we see that substituting $\widehat{\Gamma}^{\alpha}_{\nu\mu} = \tilde{\Gamma}^{\alpha}_{\nu\mu} + [\,\delta^{\alpha}_{\mu} A_{\nu} - \delta^{\alpha}_{\nu} A_{\mu}]\sqrt{2}\, i\Lambda_b^{1/2}$ from (2.6) into (2.5) gives (2.9),

$$\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) \;=\; \mathcal{R}_{\nu\mu}(\tilde{\Gamma}) + 2A_{[\nu,\mu]}\sqrt{2}\, i\Lambda_b^{1/2}. \tag{R.17}$$

Appendix S

Calculation of the non-symmetric Ricci tensor in tetrad form

Here we derive the tetrad form of the non-symmetric Ricci tensor (2.11), which is needed to calculate the Newman-Penrose asymptotically flat $\mathcal{O}(1/r^2)$ expansion of the field equations in §6.2. Let us define the "ordinary Riemann tensor" for a non-symmetric connection to be

$$\tilde{R}^{\lambda}_{\beta\sigma\tau} = \tilde{\Gamma}^{\lambda}_{\beta\tau,\sigma} - \tilde{\Gamma}^{\lambda}_{\beta\sigma,\tau} + \tilde{\Gamma}^{\nu}_{\beta\tau}\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \tilde{\Gamma}^{\nu}_{\beta\sigma}\tilde{\Gamma}^{\lambda}_{\nu\tau}. \tag{S.1}$$

For this appendix only we define the covariant derivative ";" to have the derivative index on the right side of the nonsymmetric connection $\tilde{\Gamma}^{\alpha}_{\nu\mu}$. Then we have

$$\begin{aligned}
\xi_{\beta;\sigma;\tau} - \xi_{\beta;\tau;\sigma} &= (\xi_{\beta;\sigma,\tau} - \tilde{\Gamma}^{\nu}_{\beta\tau}\xi_{\nu;\sigma} - \tilde{\Gamma}^{\nu}_{\sigma\tau}\xi_{\beta;\nu}) \\
&- (\xi_{\beta;\tau,\sigma} - \tilde{\Gamma}^{\nu}_{\beta\sigma}\xi_{\nu;\tau} - \tilde{\Gamma}^{\nu}_{\tau\sigma}\xi_{\beta;\nu}) \qquad \text{(S.2)} \\
&= [(\xi_{\beta,\sigma} - \tilde{\Gamma}^{\lambda}_{\beta\sigma}\xi_{\lambda})_{,\tau} - \tilde{\Gamma}^{\nu}_{\beta\tau}(\xi_{\lambda,\sigma} - \tilde{\Gamma}^{\lambda}_{\nu\sigma}\xi_{\lambda})] - 2\tilde{\Gamma}^{\nu}_{[\sigma\tau]}\xi_{\beta;\nu} \\
&- [(\xi_{\beta,\tau} - \tilde{\Gamma}^{\lambda}_{\beta\tau}\xi_{\lambda})_{,\sigma} - \tilde{\Gamma}^{\nu}_{\beta\sigma}(\xi_{\lambda,\tau} - \tilde{\Gamma}^{\lambda}_{\nu\sigma}\xi_{\lambda})] \qquad \text{(S.3)}
\end{aligned}$$

$$= -\tilde\Gamma^\lambda_{\beta\sigma,\tau}\xi_\lambda + \tilde\Gamma^\nu_{\beta\tau}\tilde\Gamma^\lambda_{\nu\sigma}\xi_\lambda + \tilde\Gamma^\lambda_{\beta\tau,\sigma}\xi_\lambda - \tilde\Gamma^\nu_{\beta\sigma}\tilde\Gamma^\lambda_{\nu\tau}\xi_\lambda - 2\tilde\Gamma^\nu_{[\sigma\tau]}\xi_{\beta;\nu} \quad (S.4)$$

$$= (\tilde\Gamma^\lambda_{\beta\tau,\sigma} - \tilde\Gamma^\lambda_{\beta\sigma,\tau} + \tilde\Gamma^\nu_{\beta\tau}\tilde\Gamma^\lambda_{\nu\sigma} - \tilde\Gamma^\nu_{\beta\sigma}\tilde\Gamma^\lambda_{\nu\tau})\xi_\lambda - 2\tilde\Gamma^\nu_{[\sigma\tau]}\xi_{\beta;\nu} \quad (S.5)$$

$$= \tilde R^\lambda_{\beta\sigma\tau}\xi_\lambda - 2\tilde\Gamma^\nu_{[\sigma\tau]}\xi_{\beta;\nu}. \quad (S.6)$$

Let us also define the "non-antisymmetric spin-coefficients" to be

$$\tilde\gamma_{abc} = -e_{a\sigma;c}e_b{}^\sigma = -(e_{a\sigma,c} - \tilde\Gamma^\alpha_{\sigma c}e_{a\alpha})e_b{}^\sigma = -e_{a\sigma,c}e_b{}^\sigma + \tilde\Gamma_{abc} \quad (S.7)$$

$$= \gamma_{abc} + \Upsilon_{abc}, \quad (S.8)$$

where we have made use of $\tilde\Gamma^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} + \Upsilon^\alpha_{\nu\mu}$ from (2.61). The γ_{abc} are the ordinary antisymmetric spin-coefficients formed from the Christoffel connection, meaning that $\gamma_{bac} = -\gamma_{abc}$. For the source-free case ($j^\nu = 0$) we have $\tilde\Gamma^\alpha_{\alpha[\nu,\mu]} = 0$ from (2.58), and using $\tilde\Gamma^\alpha_{[\nu\mu]} = 0$ from (2.8) we see that the non-symmetric Ricci tensor (2.11) is equivalent to the contraction of the ordinary Riemann tensor (S.1). Then from (S.6,S.8) we have

$$\tilde R^\nu_{\beta\sigma\tau}e_{a\nu} = 2e_{a\beta;[\sigma;\tau]} + 2\tilde\Gamma^\nu_{[\sigma\tau]}e_{a\beta;\nu} \quad (S.9)$$

$$= -2(e^h{}_\beta\tilde\gamma_{ahf}e^f{}_{[\sigma});_{\tau]} - 2\tilde\Gamma^\nu_{[\sigma\tau]}e^h{}_\beta\tilde\gamma_{ahf}e^f{}_\nu \quad (S.10)$$

$$= 2(e^d{}_\beta\tilde\gamma^h{}_{dg}e^g{}_{[\tau})\tilde\gamma_{ahf}e^f{}_{\sigma]} + 2e^h{}_\beta\tilde\gamma_{ahf}(e^d{}_{[\sigma}\tilde\gamma^f{}_{dg}e^g{}_{\tau]})$$

$$-e^h{}_\beta\tilde\gamma_{ahf,[\tau}e^f{}_{\sigma]} - 2\tilde\Gamma^f_{[\sigma\tau]}e^h{}_\beta\tilde\gamma_{ahf}, \quad (S.11)$$

$$\tilde R_{abcd} = 2\tilde\gamma^h{}_{b[d}\tilde\gamma_{ahc]} + 2\tilde\gamma_{abf}\tilde\gamma^f{}_{[cd]} - 2\tilde\gamma_{ab[c,d]} - 2\tilde\Gamma^f_{[cd]}\tilde\gamma_{abf} \quad (S.12)$$

$$= \tilde\gamma^h{}_{bd}\tilde\gamma_{ahc} - \tilde\gamma^h{}_{bc}\tilde\gamma_{ahd} + \tilde\gamma_{abf}\tilde\gamma^f{}_{cd} - \tilde\gamma_{abf}\tilde\gamma^f{}_{dc}$$

$$-\tilde\gamma_{abc,d} + \tilde\gamma_{abd,c} - 2\tilde\Gamma^f_{[cd]}\tilde\gamma_{abf}, \quad (S.13)$$

$$\tilde R_{bd} = \tilde\gamma^h{}_{bd}\tilde\gamma^c{}_{hc} - \tilde\gamma^h{}_{bc}\tilde\gamma^c{}_{hd} + \tilde\gamma^c{}_{bf}\tilde\gamma^f{}_{cd} - \tilde\gamma^c{}_{bf}\tilde\gamma^f{}_{dc}$$

$$-\tilde\gamma^c{}_{bc,d} + \tilde\gamma^c{}_{bd,c} - 2\tilde\Gamma^f_{[cd]}\tilde\gamma^c{}_{bf} \quad (S.14)$$

$$= \tilde\gamma^c{}_{bd,c} - \tilde\gamma^c{}_{bc,d} + \tilde\gamma^h{}_{bd}\tilde\gamma^c{}_{hc} - \tilde\gamma^c{}_{bf}(\tilde\gamma^f{}_{cd} - 2\gamma^f{}_{[cd]}). \quad (S.15)$$

For computational purposes this can be simplified a bit. Let us define

$$\tilde{\tilde{\gamma}}^c{}_{bd} \;=\; \tilde{\gamma}^c{}_{bd} - \tilde{\gamma}^g{}_{bg}\delta^c_d \tag{S.16}$$

so that

$$\tilde{\gamma}^c{}_{bd} \;=\; \tilde{\tilde{\gamma}}^c{}_{bd} - \frac{1}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\delta^c_d, \quad \tilde{\tilde{\gamma}}^c{}_{bc} = -3\tilde{\gamma}^c{}_{bc}, \quad \tilde{\tilde{\gamma}}^c{}_{cb} = 2\tilde{\gamma}^c{}_{[cb]}. \tag{S.17}$$

Then the Ricci tensor becomes

$$\begin{aligned}
\tilde{R}_{bd} \;=\;& \tilde{\tilde{\gamma}}^c{}_{bd,c} + \left(\tilde{\tilde{\gamma}}^h{}_{bd} - \frac{1}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\delta^h_d\right)\left(\tilde{\tilde{\gamma}}^c{}_{hc} - \frac{1}{3}\tilde{\tilde{\gamma}}^e{}_{he}4\right)\\
&- \left(\tilde{\tilde{\gamma}}^c{}_{bf} - \frac{1}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\delta^c_f\right)\left(\tilde{\tilde{\gamma}}^f{}_{cd} - \frac{1}{3}\tilde{\tilde{\gamma}}^e{}_{ce}\delta^f_d\right)\\
&+ 2\left(\tilde{\tilde{\gamma}}^c{}_{bf} - \frac{1}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\delta^c_f\right)\gamma^f{}_{[cd]} && \text{(S.18)}\\
=\;& \tilde{\tilde{\gamma}}^c{}_{bd,c} - \tilde{\tilde{\gamma}}^c{}_{bf}\tilde{\tilde{\gamma}}^f{}_{cd} - \frac{1}{9}\tilde{\tilde{\gamma}}^g{}_{bg}\tilde{\tilde{\gamma}}^e{}_{de} + \frac{1}{9}\tilde{\tilde{\gamma}}^g{}_{bg}\tilde{\tilde{\gamma}}^c{}_{dc} + \frac{1}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\tilde{\tilde{\gamma}}^f{}_{fd}\\
&+ 2\left(\tilde{\tilde{\gamma}}^c{}_{bf} - \frac{1}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\delta^c_f\right)\gamma^f{}_{[cd]} && \text{(S.19)}\\
=\;& \tilde{\tilde{\gamma}}^c{}_{bd,c} - \tilde{\tilde{\gamma}}^c{}_{bf}\tilde{\tilde{\gamma}}^f{}_{cd} + 2\tilde{\tilde{\gamma}}^c{}_{bf}\gamma^f{}_{[cd]} - \frac{2}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\left(\gamma^f{}_{[fd]} - \tilde{\gamma}^f{}_{[fd]}\right) && \text{(S.20)}\\
=\;& \tilde{\tilde{\gamma}}^c{}_{bd,c} - \tilde{\tilde{\gamma}}^c{}_{bf}\tilde{\tilde{\gamma}}^f{}_{cd} + 2\tilde{\tilde{\gamma}}^c{}_{bf}\gamma^f{}_{[cd]} + \frac{2}{3}\tilde{\tilde{\gamma}}^g{}_{bg}\Upsilon^f_{[fd]} && \text{(S.21)}\\
=\;& \tilde{\tilde{\gamma}}^c{}_{bd,c} + 2\tilde{\tilde{\gamma}}^c{}_{bf}(2\gamma^f{}_{[cd]} - \tilde{\tilde{\gamma}}^f{}_{cd}). && \text{(S.22)}
\end{aligned}$$

In (S.21), $\Upsilon^f_{[fd]} = 0$ because $\tilde{\Gamma}^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} + \Upsilon^\alpha_{\nu\mu}$ from (2.61) and $\tilde{\Gamma}^f_{[fd]} = \Gamma^f_{[fd]} = 0$ from (2.8,2.20).

Appendix T

Proof of a nonsymmetric matrix decomposition theorem

Here we prove the theorem (6.3-6.12). **Proof**: From [64] p.51, any antisymmetric real tensor f_{ab} as in (6.10,6.12) can be parameterized in Newman-Penrose form by the three complex scalars

$$\phi_0 = \hat{f}_{13} \quad , \quad \phi_1 = (\hat{f}_{12} + \hat{f}_{43})/2 \quad , \quad \phi_2 = \hat{f}_{42}. \tag{T.1}$$

From [64] p.53-54, there are three tetrad transformations which do not alter (6.3).

Type I:

$$l_\sigma \;\rightarrow\; l_\sigma, \;\; m_\sigma \rightarrow m_\sigma{+}al_\sigma, \;\; m^\star_\sigma \rightarrow m^\star_\sigma{+}a^\star l_\sigma, \;\; n_\sigma \rightarrow n_\sigma{+}a^\star m_\sigma{+}am^\star_\sigma{+}aa^\star l_\sigma, \tag{T.2}$$

$$\phi_0 \;\rightarrow\; \phi_0, \;\; \phi_1 \rightarrow \phi_1{+}a^\star\phi_0, \;\; \phi_2 \rightarrow \phi_2{+}2a^\star\phi_1{+}(a^\star)^2\phi_0. \tag{T.3}$$

Type II:

$$n_\sigma \;\rightarrow\; n_\sigma, \;\; m_\sigma \rightarrow m_\sigma{+}bn_\sigma, \;\; m^\star_\sigma \rightarrow m^\star_\sigma{+}b^\star n_\sigma, \;\; l_\sigma \rightarrow l_\sigma{+}b^\star m_\sigma{+}bm^\star_\sigma{+}bb^\star n_\sigma, \tag{T.4}$$

$$\phi_2 \;\rightarrow\; \phi_2, \;\; \phi_1 \rightarrow \phi_1{+}b\phi_2, \;\; \phi_0 \rightarrow \phi_0{+}2b\phi_1{+}b^2\phi_2. \tag{T.5}$$

Type III:

$$n_\sigma \;\rightarrow\; n_\sigma A,\quad l_\sigma \rightarrow l_\sigma/A,\quad m_\sigma \rightarrow m_\sigma e^{i\theta},\quad m^\star_\sigma \rightarrow m^\star_\sigma e^{-i\theta}, \tag{T.6}$$

$$\phi_1 \;\rightarrow\; \phi_1,\quad \phi_0 \rightarrow \phi_0 e^{i\theta}/A,\quad \phi_2 \rightarrow \phi_2 e^{-i\theta} A. \tag{T.7}$$

Using type I and II transformations, we can always make either $\phi_2 = 0$ or $\phi_0 = 0$ by solving a quadradic equation and performing a tetrad transformation with

$$a^\star = \frac{-2\phi_1 \pm \sqrt{(2\phi_1)^2 - 4\phi_0\phi_2}}{2\phi_0} \qquad \text{or} \qquad b = \frac{-2\phi_1 \pm \sqrt{(2\phi_1)^2 - 4\phi_2\phi_0}}{2\phi_2}. \tag{T.8}$$

Note that a type I transformation does not alter ϕ_0 and a type II transformation does not alter ϕ_2. Therefore, if $\phi_1 \neq 0$ at this point, we can make $\phi_0 = \phi_2 = 0$ by doing a second transformation of the opposite type to the first one with

$$b = -\frac{\phi_0}{2\phi_1} \qquad \text{or} \qquad a^\star = -\frac{\phi_2}{2\phi_1}. \tag{T.9}$$

Then with $\breve{u} = -2Re(\phi_1)$, $\grave{u} = -2Im(\phi_1)$, we get from (T.1,6.10) the first case (6.3,6.9). The procedure above fails if $\phi_1 = 0$ in (T.9), in which case there is only one nonzero scalar, either ϕ_0 or ϕ_2. If the nonzero scalar is ϕ_2, it can be changed to ϕ_0 by doing type II transformation with $b = 1$ followed by a type I transformation with $a^* = -1$. Furthermore, we can make ϕ_0 real by doing a type III transformation. Then with $\acute{u} = \phi_0$ we get from (T.1,6.12) the second case (6.3,6.11). Since $\hat{f}^\sigma{}_\mu \hat{f}^\mu{}_\sigma = \hat{f}^a{}_b \hat{f}^b{}_a$ and $det(\hat{f}^\mu{}_\nu) = det(\hat{f}^a{}_b)$, we see from (6.10,6.12) that this second case occurs if and only if $\hat{f}^\sigma{}_\mu \hat{f}^\mu{}_\sigma = det(\hat{f}^\mu{}_\nu) = 0$. If we change $\grave{u}$ and $\breve{u}$ from real to imaginary and do not change the tetrads, $g^{\nu\mu}$ will stay real and $f^{\nu\mu}$ will become imaginary, and therefore $W^{\nu\mu}$ becomes Hermitian. This proves the theorem.

Appendix U

Calculation of the exact Υ^a_{bc} in Newman-Penrose form

Here we solve the connection equations (2.59) in tetrad form (6.54-6.64) to give the result (6.66-6.95). In the following linear combinations of the connection equations (6.54-6.64), the right-hand-side Υ^a_{bc} terms cancel,

$$
\begin{aligned}
{}^{\pm}\Upsilon^2_{12} &= {}^{\pm}\Upsilon^2_{12} + \frac{1}{2}\left({}^{\pm}O^{12}_1 - O^{22}_2 - {}^{\pm}O^{21}_1 \frac{(1\mp\breve{u})}{(1\pm\breve{u})}\right), && \text{(U.1)}\\
{}^{\pm}\Upsilon^1_{12} &= {}^{\pm}\Upsilon^1_{12} + \frac{1}{2}\left({}^{\pm}O^{12}_2 - O^{11}_1 - {}^{\pm}O^{21}_2 \frac{(1\mp\breve{u})}{(1\pm\breve{u})}\right), && \text{(U.2)}\\
{}^{\pm}\Upsilon^4_{34} &= {}^{\pm}\Upsilon^4_{34} + \frac{1}{2}\left(-{}^{\pm}O^{34}_3 + O^{44}_4 + {}^{\pm}O^{43}_3 \frac{(1\mp i\grave{u})}{(1\pm i\grave{u})}\right), && \text{(U.3)}\\
\Upsilon^1_{11} &= \Upsilon^1_{11} + \frac{1}{2}(-O^{21}_1 - O^{12}_1 + O^{22}_2), && \text{(U.4)}\\
\Upsilon^2_{22} &= \Upsilon^2_{22} + \frac{1}{2}(-O^{12}_2 - O^{21}_2 + O^{11}_1), && \text{(U.5)}\\
\Upsilon^3_{33} &= \Upsilon^3_{33} + \frac{1}{2}(O^{43}_3 + O^{34}_3 - O^{44}_4), && \text{(U.6)}\\
\Upsilon^2_{11} &= \Upsilon^2_{11} - \frac{1}{2}O^{22}_1, && \text{(U.7)}\\
\Upsilon^1_{22} &= \Upsilon^1_{22} - \frac{1}{2}O^{11}_2, && \text{(U.8)}
\end{aligned}
$$

$$\Upsilon^3_{44} = \Upsilon^3_{44} + \frac{1}{2} O^{33}_4, \tag{U.9}$$

$$^{\pm}\Upsilon^2_{23} = {}^{\pm}\Upsilon^2_{23} + \frac{1}{2} \left(\ {}^{\pm}O^{24}_2 - {}^{\pm}O^{41}_1 - {}^{\pm}O^{12}_3 \frac{(1 \pm i\grave{u})}{(1 \mp \check{u})} \right), \tag{U.10}$$

$$^{\pm}\Upsilon^1_{13} = {}^{\pm}\Upsilon^1_{13} + \frac{1}{2} \left(-{}^{\pm}O^{42}_2 + {}^{\pm}O^{14}_1 - {}^{\pm}O^{21}_3 \frac{(1 \pm i\grave{u})}{(1 \pm \check{u})} \right), \tag{U.11}$$

$$^{\pm}\Upsilon^3_{13} = {}^{\pm}\Upsilon^3_{13} + \frac{1}{2} \left(\ {}^{\pm}O^{42}_4 - {}^{\pm}O^{23}_3 + {}^{\pm}O^{34}_1 \frac{(1 \mp \check{u})}{(1 \mp i\grave{u})} \right), \tag{U.12}$$

$$^{\pm}\Upsilon^3_{23} = {}^{\pm}\Upsilon^3_{23} + \frac{1}{2} \left(\ {}^{\pm}O^{41}_4 - {}^{\pm}O^{13}_3 + {}^{\pm}O^{34}_2 \frac{(1 \pm \check{u})}{(1 \mp i\grave{u})} \right), \tag{U.13}$$

$$^{\pm}\Upsilon^4_{12} = {}^{\pm}\Upsilon^4_{12} + \frac{1}{2} \left(-{}^{\pm}O^{24}_2 \frac{(1 \pm i\grave{u})}{(1 \pm \check{u})} - {}^{\pm}O^{41}_1 \frac{(1 \mp i\grave{u})}{(1 \pm \check{u})} - {}^{\pm}O^{12}_3 \frac{(1 + \grave{u}^2)}{(1 - \check{u}^2)} \right), \tag{U.14}$$

$$^{\pm}\Upsilon^2_{34} = {}^{\pm}\Upsilon^2_{34} + \frac{1}{2} \left(\ {}^{\pm}O^{42}_4 \frac{(1 \pm \check{u})}{(1 \pm i\grave{u})} + {}^{\pm}O^{23}_3 \frac{(1 \mp \check{u})}{(1 \pm i\grave{u})} + {}^{\pm}O^{34}_1 \frac{(1 - \check{u}^2)}{(1 + \grave{u}^2)} \right), \tag{U.15}$$

$$^{\pm}\Upsilon^1_{43} = {}^{\pm}\Upsilon^1_{43} + \frac{1}{2} \left(\ {}^{\pm}O^{31}_3 \frac{(1 \mp \check{u})}{(1 \mp i\grave{u})} + {}^{\pm}O^{14}_4 \frac{(1 \pm \check{u})}{(1 \mp i\grave{u})} + {}^{\pm}O^{43}_2 \frac{(1 - \check{u}^2)}{(1 + \grave{u}^2)} \right), \tag{U.16}$$

$$^{\pm}\Upsilon^2_{13} = {}^{\pm}\Upsilon^2_{13} + \frac{1}{2\check{z}} \left(-{}^{\pm}O^{42}_1 (1 - \check{u}^2) + {}^{\pm}O^{24}_1 (1 \mp \check{u})^2 - O^{22}_3 (1 \pm i\grave{u})(1 \pm \check{u}) \right), \tag{U.17}$$

$$^{\pm}\Upsilon^1_{24} = {}^{\pm}\Upsilon^1_{24} + \frac{1}{2\check{z}} \left(-{}^{\pm}O^{31}_2 (1 - \check{u}^2) + {}^{\pm}O^{13}_2 (1 \pm \check{u})^2 - O^{11}_4 (1 \mp i\grave{u})(1 \mp \check{u}) \right), \tag{U.18}$$

$$^{\pm}\Upsilon^4_{13} = {}^{\pm}\Upsilon^4_{13} + \frac{1}{2\grave{z}} \left(\ {}^{\pm}O^{42}_3 (1 + \grave{u}^2) - {}^{\pm}O^{24}_3 (1 \pm i\grave{u})^2 + O^{44}_1 (1 \mp i\grave{u})(1 \mp \check{u}) \right), \tag{U.19}$$

$$^{\pm}\Upsilon^3_{24} = {}^{\pm}\Upsilon^3_{24} + \frac{1}{2\grave{z}} \left(\ {}^{\pm}O^{31}_4 (1 + \grave{u}^2) - {}^{\pm}O^{13}_4 (1 \mp i\grave{u})^2 + O^{33}_2 (1 \pm i\grave{u})(1 \pm \check{u}) \right), \tag{U.20}$$

$$\Upsilon^4_{11} = \Upsilon^4_{11} + \frac{1}{2\check{z}} \left(-O^{24}_1 (1 + i\grave{u})(1 + \check{u}) - O^{42}_1 (1 - i\grave{u})(1 - \check{u}) - O^{22}_3 (1 + \grave{u}^2) \right), \tag{U.21}$$

$$\Upsilon^3_{22} = \Upsilon^3_{22} + \frac{1}{2\check{z}} \left(-O^{13}_2 (1 - i\grave{u})(1 - \check{u}) - O^{31}_2 (1 + i\grave{u})(1 + \check{u}) - O^{11}_4 (1 + \grave{u}^2) \right), \tag{U.22}$$

$$\Upsilon^2_{33} = \Upsilon^2_{33} + \frac{1}{2\grave{z}} \left(\ O^{42}_3 (1 + i\grave{u})(1 + \check{u}) + O^{24}_3 (1 - i\grave{u})(1 - \check{u}) + O^{44}_1 (1 - \check{u}^2) \right), \tag{U.23}$$

$$\Upsilon^1_{44} = \Upsilon^1_{44} + \frac{1}{2\grave{z}} \left(\ O^{31}_4 (1 - i\grave{u})(1 - \check{u}) + O^{13}_4 (1 + i\grave{u})(1 + \check{u}) + O^{33}_2 (1 - \check{u}^2) \right). \tag{U.24}$$

Performing the linear combinations above using (6.45-6.51,6.65) gives

$$
\begin{aligned}
{}^{\pm}\Upsilon^2_{12} &= \mp\frac{D\breve{u}}{(1\pm\breve{u})} \pm \frac{4\pi}{3c(1\pm\breve{u})}\hat{j}^2, && \text{(U.25)}\\
{}^{\pm}\Upsilon^1_{12} &= \mp\frac{\Delta\breve{u}}{(1\pm\breve{u})} \mp \frac{4\pi}{3c(1\pm\breve{u})}\hat{j}^1, && \text{(U.26)}\\
{}^{\pm}\Upsilon^4_{34} &= \mp\frac{i\delta\grave{u}}{(1\pm i\grave{u})} \mp \frac{4\pi}{3c(1\pm i\grave{u})}\hat{j}^4, && \text{(U.27)}\\
\Upsilon^1_{11} &= \frac{D\sqrt{-N_\diamond}}{\sqrt{-N_\diamond}} + \frac{4\pi\breve{u}\breve{c}^2}{3c}\hat{j}^2 = \grave{u}D\grave{u}\grave{c}^2 - \breve{u}D\breve{u}\breve{c}^2 + \frac{4\pi\breve{u}\breve{c}^2}{3c}\hat{j}^2, && \text{(U.28)}\\
\Upsilon^2_{22} &= \frac{\Delta\sqrt{-N_\diamond}}{\sqrt{-N_\diamond}} - \frac{4\pi\breve{u}\breve{c}^2}{3c}\hat{j}^1 = \grave{u}\Delta\grave{u}\grave{c}^2 - \breve{u}\Delta\breve{u}\breve{c}^2 - \frac{4\pi\breve{u}\breve{c}^2}{3c}\hat{j}^1, && \text{(U.29)}\\
\Upsilon^3_{33} &= \frac{\delta\sqrt{-N_\diamond}}{\sqrt{-N_\diamond}} - \frac{4\pi i\grave{u}\grave{c}^2}{3c}\hat{j}^4 = \grave{u}\delta\grave{u}\grave{c}^2 - \breve{u}\delta\breve{u}\breve{c}^2 - \frac{4\pi i\grave{u}\grave{c}^2}{3c}\hat{j}^4, && \text{(U.30)}\\
\Upsilon^2_{11} &= \Upsilon^1_{22} = \Upsilon^3_{44} = 0, && \text{(U.31)}\\
{}^{\pm}\Upsilon^2_{23} &= \mp\frac{1}{2}(\tau w + \pi^* w^*) - \frac{1}{2}(\mp\delta\breve{u}\breve{c}^2 - \grave{u}\delta\grave{u}\grave{c}^2)(1\pm i\grave{u}) \pm \frac{2\pi(2\mp 3i\grave{u})}{3c(1\mp i\grave{u})}\hat{j}^4 && \text{(U.32)}\\
&= \mp\frac{1}{2}(i\delta\grave{u}\grave{c}^2(1\mp i\grave{u}) - \delta\breve{u}\breve{c}^2 \mp \grave{u}\delta\grave{u}\grave{c}^2)(1\pm i\grave{u}) \mp \frac{2\pi}{3c(1\mp i\grave{u})}\hat{j}^4 && \text{(U.33)}\\
&= \pm\frac{1}{2}(\delta\breve{u}\breve{c}^2 - i\delta\grave{u}\grave{c}^2)(1\pm i\grave{u}) \mp \frac{2\pi}{3c(1\mp i\grave{u})}\hat{j}^4, && \text{(U.34)}\\
{}^{\pm}\Upsilon^1_{13} &= \mp\frac{1}{2}(\tau w + \pi^* w^*) - \frac{1}{2}(\pm\delta\breve{u}\breve{c}^2 - \grave{u}\delta\grave{u}\grave{c}^2)(1\pm i\grave{u}) \pm \frac{2\pi(2\mp 3i\grave{u})}{3c(1\mp i\grave{u})}\hat{j}^4 && \text{(U.35)}\\
&= \mp\frac{1}{2}(i\delta\grave{u}\grave{c}^2(1\mp i\grave{u}) + \delta\breve{u}\breve{c}^2 \mp \grave{u}\delta\grave{u}\grave{c}^2)(1\pm i\grave{u}) \mp \frac{2\pi}{3c(1\mp i\grave{u})}\hat{j}^4 && \text{(U.36)}\\
&= \mp\frac{1}{2}(\delta\breve{u}\breve{c}^2 + i\delta\grave{u}\grave{c}^2)(1\pm i\grave{u}) \mp \frac{2\pi}{3c(1\mp i\grave{u})}\hat{j}^4, && \text{(U.37)}\\
{}^{\pm}\Upsilon^3_{13} &= \pm\frac{1}{2}(\rho w + \rho^* w^*) + \frac{1}{2}(\pm iD\grave{u}\grave{c}^2 - \breve{u}D\breve{u}\breve{c}^2)(1\mp\breve{u}) \pm \frac{2\pi(2\pm 3\breve{u})}{3c(1\pm\breve{u})}\hat{j}^2 && \text{(U.38)}\\
&= \pm\frac{1}{2}(D\breve{u}\breve{c}^2(1\pm\breve{u}) + iD\grave{u}\grave{c}^2 \mp \breve{u}D\breve{u}\breve{c}^2)(1\mp\breve{u}) \mp \frac{2\pi}{3c(1\pm\breve{u})}\hat{j}^2 && \text{(U.39)}\\
&= \pm\frac{1}{2}(D\breve{u}\breve{c}^2 + iD\grave{u}\grave{c}^2)(1\mp\breve{u}) \mp \frac{2\pi}{3c(1\pm\breve{u})}\hat{j}^2, && \text{(U.40)}\\
{}^{\pm}\Upsilon^3_{23} &= \pm\frac{1}{2}(\mu w + \mu^* w^*) + \frac{1}{2}(\pm i\Delta\grave{u}\grave{c}^2 - \breve{u}\Delta\breve{u}\breve{c}^2)(1\pm\breve{u}) \pm \frac{2\pi(2\mp 3\breve{u})}{3c(1\mp\breve{u})}\hat{j}^1 && \text{(U.41)}\\
&= \pm\frac{1}{2}(-\Delta\breve{u}\breve{c}^2(1\mp\breve{u}) + i\Delta\grave{u}\grave{c}^2 \mp \breve{u}\Delta\breve{u}\breve{c}^2)(1\pm\breve{u}) \mp \frac{2\pi}{3c(1\mp\breve{u})}\hat{j}^1 && \text{(U.42)}\\
&= \mp\frac{1}{2}(\Delta\breve{u}\breve{c}^2 - i\Delta\grave{u}\grave{c}^2)(1\pm\breve{u}) \mp \frac{2\pi}{3c(1\mp\breve{u})}\hat{j}^1, && \text{(U.43)}
\end{aligned}
$$

$$
\begin{aligned}
{}^{\pm}\Upsilon^{4}_{12} &= \frac{1}{2(1\pm\check{u})}\left(\pm\tau w(1\pm i\grave{u}) \mp \pi^{*}w^{*}(1\mp i\grave{u}) \pm \delta\check{u}\frac{\check{c}^{2}}{\grave{c}^{2}} + \grave{u}\delta\grave{u} - \frac{4\pi}{c}i\grave{u}\hat{j}^{4}\right) && \text{(U.44)}\\
&= \frac{\pm 1}{2(1\pm\check{u})}\left(\delta\check{u}\frac{\check{c}^{2}}{\grave{c}^{2}} + \tau w - \pi^{*}w^{*}\right), && \text{(U.45)}\\
{}^{\pm}\Upsilon^{2}_{34} &= \frac{1}{2(1\pm i\grave{u})}\left(\pm\rho w(1\pm\check{u}) \mp \rho^{*}w^{*}(1\mp\check{u}) \pm iD\grave{u}\frac{\grave{c}^{2}}{\check{c}^{2}} - \check{u}D\check{u} + \frac{4\pi}{c}\check{u}\hat{j}^{2}\right) && \text{(U.46)}\\
&= \frac{\pm 1}{2(1\pm i\grave{u})}\left(iD\grave{u}\frac{\grave{c}^{2}}{\check{c}^{2}} + \rho w - \rho^{*}w^{*}\right), && \text{(U.47)}\\
{}^{\pm}\Upsilon^{1}_{43} &= \frac{1}{2(1\mp i\grave{u})}\left(\pm\mu w(1\mp\check{u}) \mp \mu^{*}w^{*}(1\pm\check{u}) \mp i\Delta\grave{u}\frac{\grave{c}^{2}}{\check{c}^{2}} - \check{u}\Delta\check{u} - \frac{4\pi}{c}\check{u}\hat{j}^{1}\right) && \text{(U.48)}\\
&= \frac{\mp 1}{2(1\mp i\grave{u})}\left(i\Delta\grave{u}\frac{\grave{c}^{2}}{\check{c}^{2}} - \mu w + \mu^{*}w^{*}\right), && \text{(U.49)}\\
{}^{\pm}\Upsilon^{2}_{13} &= \frac{\kappa w(\check{u}\mp 1)}{\check{z}} \quad , \quad {}^{\pm}\Upsilon^{1}_{24} = -\frac{\nu w(\check{u}\pm 1)}{\check{z}}, && \text{(U.50)}\\
{}^{\pm}\Upsilon^{4}_{13} &= \frac{\sigma w(i\grave{u}\pm 1)}{\grave{z}} \quad , \quad {}^{\pm}\Upsilon^{3}_{24} = -\frac{\lambda w(i\grave{u}\mp 1)}{\grave{z}}, && \text{(U.51)}\\
\Upsilon^{4}_{11} &= \frac{\kappa w^{2}}{\check{z}} \quad , \quad \Upsilon^{3}_{22} = -\frac{\nu w^{2}}{\check{z}}, && \text{(U.52)}\\
\Upsilon^{2}_{33} &= \frac{\sigma w^{2}}{\grave{z}} \quad , \quad \Upsilon^{1}_{44} = -\frac{\lambda w^{2}}{\grave{z}}. && \text{(U.53)}
\end{aligned}
$$

Appendix V

Check of the approximate Υ^a_{bc} in Newman-Penrose form

Here we will show that the $\mathcal{O}(\Lambda_b^{-1})$ approximation of $\Upsilon^\alpha_{\sigma\mu}$ in (2.62,2.63,2.64) matches the exact solution (6.66-6.95) for $\grave{c}=\check{c}=\grave{z}=\check{z}=1$, which amounts to a second order approximation in $\grave{u}$ and $\check{u}$. Much use is made of g_{ab} and $\hat{f}_{ab}$ from (6.3,6.10), $\gamma_{cab}=-\gamma_{acb}$ from (6.15), $\ell_{,a}/4=\check{u}\check{u}_{,a}-\grave{u}\grave{u}_{,a}$ from (6.33), and the field equations (6.45-6.51). To save space, only one component of each type will be shown.

In tetrad form (2.62) becomes,

$$
\begin{aligned}
\Upsilon_{c(de)} \;&\approx\; \frac{1}{2}(\hat{f}^a{}_d(\hat{f}_{ec,a}-\gamma^b_{ea}\hat{f}_{bc}-\gamma^b_{ca}\hat{f}_{eb})+\hat{f}^a{}_e(\hat{f}_{dc,a}-\gamma^b_{da}\hat{f}_{bc}-\gamma^b_{ca}\hat{f}_{db})\\
&\quad+\hat{f}_c{}^a(\hat{f}_{ad,e}-\gamma^b_{ae}\hat{f}_{bd}-\gamma^b_{de}\hat{f}_{ab})+\hat{f}_c{}^a(\hat{f}_{ae,d}-\gamma^b_{ad}\hat{f}_{be}-\gamma^b_{ed}\hat{f}_{ab}))\\
&\quad+\frac{1}{8}(\,\ell_{,c}g_{de}-\,\ell_{,d}g_{ec}-\,\ell_{,e}g_{dc})\\
&\quad+\frac{2\pi}{c}\left(\hat{j}^a\hat{f}_{ca}g_{de}+\frac{1}{3}\hat{j}^a\hat{f}_{ad}g_{ec}+\frac{1}{3}\hat{j}^a\hat{f}_{ae}g_{dc}\right), \qquad \text{(V.1)}\\
\Upsilon_{1(12)} \;&\approx\; \frac{1}{2}(\hat{f}^a{}_1(\hat{f}_{21,a}-\gamma^b_{2a}\hat{f}_{b1}-\gamma^b_{1a}\hat{f}_{2b})+\hat{f}^a{}_2(\hat{f}_{11,a}-\gamma^b_{1a}\hat{f}_{b1}-\gamma^b_{1a}\hat{f}_{1b})
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\hat{f}_1{}^a(\hat{f}_{a1,2}-\gamma^b_{a2}\hat{f}_{b1}-\gamma^b_{12}\hat{f}_{ab})+\hat{f}_1{}^a(\hat{f}_{a2,1}-\gamma^b_{a1}\hat{f}_{b2}-\gamma^b_{21}\hat{f}_{ab}))\\
&\quad +\frac{1}{8}(\ \ell_{,1}g_{12}-\ \ell_{,1}g_{21}-\ \ell_{,2}g_{11})\\
&\quad +\frac{2\pi}{c}\left(\hat{j}^2\hat{f}_{12}g_{12}+\frac{1}{3}\hat{j}^2\hat{f}_{21}g_{21}+\frac{1}{3}\hat{j}^1\hat{f}_{12}g_{11}\right) \qquad \text{(V.2)}\\
&= \breve{u}D\breve{u}-\frac{4\pi}{3c}\breve{u}\hat{j}^2, \qquad \text{(V.3)}
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{2(11)} &\approx \quad \hat{f}^a{}_1(\hat{f}_{12,a}-\gamma^b_{1a}\hat{f}_{b2}-\gamma^b_{2a}\hat{f}_{1b})+\hat{f}_2{}^a(\hat{f}_{a1,1}-\gamma^b_{a1}\hat{f}_{b1}-\gamma^b_{11}\hat{f}_{ab})\\
&\quad +\frac{1}{8}(\ \ell_{,2}g_{11}-\ \ell_{,1}g_{12}-\ \ell_{,1}g_{12})\\
&\quad +\frac{2\pi}{c}\left(\hat{j}^1\hat{f}_{21}g_{11}+\frac{1}{3}\hat{j}^2\hat{f}_{21}g_{12}+\frac{1}{3}\hat{j}^2\hat{f}_{21}g_{12}\right) \qquad \text{(V.4)}\\
&= \grave{u}D\grave{u}-\breve{u}D\breve{u}+\frac{4\pi}{3c}\breve{u}\hat{j}^2, \qquad \text{(V.5)}
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{1(11)} &\approx \quad \hat{f}^a{}_1(\hat{f}_{11,a}-\gamma^b_{1a}\hat{f}_{b1}-\gamma^b_{1a}\hat{f}_{1b})+\hat{f}_1{}^a(\hat{f}_{a1,1}-\gamma^b_{a1}\hat{f}_{b1}-\gamma^b_{11}\hat{f}_{ab})\\
&\quad +\frac{1}{8}(\ \ell_{,1}g_{11}-\ \ell_{,1}g_{11}-\ \ell_{,1}g_{11})\\
&\quad +\frac{2\pi}{c}\left(\hat{j}^2\hat{f}_{12}g_{11}+\frac{1}{3}\hat{j}^2\hat{f}_{21}g_{11}+\frac{1}{3}\hat{j}^2\hat{f}_{21}g_{11}\right) \qquad \text{(V.6)}\\
&= 0, \qquad \text{(V.7)}
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{1(23)} &\approx \frac{1}{2}(\hat{f}^a{}_2(\hat{f}_{31,a}-\gamma^b_{3a}\hat{f}_{b1}-\gamma^b_{1a}\hat{f}_{3b})+\hat{f}^a{}_3(\hat{f}_{21,a}-\gamma^b_{2a}\hat{f}_{b1}-\gamma^b_{1a}\hat{f}_{2b})\\
&\quad +\hat{f}_1{}^a(\hat{f}_{a2,3}-\gamma^b_{a3}\hat{f}_{b2}-\gamma^b_{23}\hat{f}_{ab})+\hat{f}_1{}^a(\hat{f}_{a3,2}-\gamma^b_{a2}\hat{f}_{b3}-\gamma^b_{32}\hat{f}_{ab}))\\
&\quad +\frac{1}{8}(\ \ell_{,1}g_{23}-\ \ell_{,2}g_{31}-\ \ell_{,3}g_{21})\\
&\quad +\frac{2\pi}{c}\left(\hat{j}^2\hat{f}_{12}g_{23}+\frac{1}{3}\hat{j}^1\hat{f}_{12}g_{31}+\frac{1}{3}\hat{j}^4\hat{f}_{43}g_{21}\right) \qquad \text{(V.8)}\\
&= \frac{1}{2}(i\grave{u}\delta\breve{u}+\breve{u}\delta\breve{u})-\frac{1}{2}(\breve{u}\delta\breve{u}-\grave{u}\delta\grave{u})-\frac{2\pi}{3c}i\grave{u}\hat{j}^4 \qquad \text{(V.9)}\\
&= \frac{i\grave{u}}{2}(\delta\breve{u}-i\delta\grave{u})-\frac{2\pi}{3c}i\grave{u}\hat{j}^4, \qquad \text{(V.10)}
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{3(12)} &\approx \frac{1}{2}(\hat{f}^a{}_1(\hat{f}_{23,a}-\gamma^b_{2a}\hat{f}_{b3}-\gamma^b_{3a}\hat{f}_{2b})+\hat{f}^a{}_2(\hat{f}_{13,a}-\gamma^b_{1a}\hat{f}_{b3}-\gamma^b_{3a}\hat{f}_{1b})\\
&\quad +\hat{f}_3{}^a(\hat{f}_{a1,2}-\gamma^b_{a2}\hat{f}_{b1}-\gamma^b_{12}\hat{f}_{ab})+\hat{f}_3{}^a(\hat{f}_{a2,1}-\gamma^b_{a1}\hat{f}_{b2}-\gamma^b_{21}\hat{f}_{ab}))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}(\,\ell_{,3}g_{12} - \,\ell_{,1}g_{23} - \,\ell_{,2}g_{13}) \\
&+\frac{2\pi}{c}\left(\hat{j}^4\hat{f}_{34}g_{12} + \frac{1}{3}\hat{j}^2\hat{f}_{21}g_{23} + \frac{1}{3}\hat{j}^1\hat{f}_{12}g_{13}\right) \qquad \text{(V.11)} \\
= \;& \frac{1}{2}\breve{u}(-\gamma_{321}i\grave{u} - \gamma_{231}\breve{u} + \gamma_{312}i\grave{u} - \gamma_{132}\breve{u}) \\
+ \;& \frac{1}{2}i\grave{u}(\gamma_{132}\breve{u} - \gamma_{312}i\grave{u} - \gamma_{231}\breve{u} - \gamma_{321}i\grave{u}) + \frac{1}{2}(\breve{u}\delta\breve{u} - \grave{u}\delta\grave{u}) + \frac{2\pi}{c}i\grave{u}\hat{j}^4 \qquad \text{(V.12)} \\
= \;& \frac{1}{2}\breve{u}(\delta\breve{u} - \pi^*\breve{u} + \tau\breve{u}) + \frac{1}{2}i\grave{u}(i\delta\grave{u} - \tau i\grave{u} + \pi^* i\grave{u}) + \frac{2\pi}{c}i\grave{u}\hat{j}^4 \qquad \text{(V.13)} \\
= \;& \frac{1}{2}\breve{u}(\delta\breve{u} - \pi^*\breve{u} + \tau\breve{u}) + \frac{1}{2}i\grave{u}(\tau w + \pi^* w^* - \tau i\grave{u} + \pi^* i\grave{u}) \qquad \text{(V.14)} \\
= \;& \frac{1}{2}\breve{u}(\delta\breve{u} - \pi^*\breve{u} + \tau\breve{u}) + \frac{1}{2}i\grave{u}(\tau\breve{u} + \pi^*\breve{u}) \qquad \text{(V.15)} \\
= \;& \frac{1}{2}\breve{u}(\delta\breve{u} - \pi^*\breve{u} + \tau\breve{u} + \tau i\grave{u} + \pi^* i\grave{u}) \qquad \text{(V.16)} \\
= \;& \frac{1}{2}\breve{u}(\delta\breve{u} + \tau w - \pi^* w^*), \qquad \text{(V.17)}
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{1(13)} \approx \;& \frac{1}{2}(\hat{f}^a{}_1(\hat{f}_{31,a} - \gamma^b_{3a}\hat{f}_{b1} - \gamma^b_{1a}\hat{f}_{3b}) + \hat{f}^a{}_3(\hat{f}_{11,a} - \gamma^b_{1a}\hat{f}_{b1} - \gamma^b_{1a}\hat{f}_{1b}) \\
&+\hat{f}_1{}^a(\hat{f}_{a1,3} - \gamma^b_{a3}\hat{f}_{b1} - \gamma^b_{13}\hat{f}_{ab}) + \hat{f}_1{}^a(\hat{f}_{a3,1} - \gamma^b_{a1}\hat{f}_{b3} - \gamma^b_{31}\hat{f}_{ab})) \\
&+\frac{1}{8}(\,\ell_{,1}g_{13} - \,\ell_{,1}g_{31} - \,\ell_{,3}g_{11}) \\
&+\frac{2\pi}{c}\left(\hat{j}^2\hat{f}_{12}g_{13} + \frac{1}{3}\hat{j}^2\hat{f}_{21}g_{31} + \frac{1}{3}\hat{j}^4\hat{f}_{43}g_{11}\right) \qquad \text{(V.18)} \\
= \;& \breve{u}(-\gamma_{131}\breve{u} + \gamma_{311}i\grave{u}) \qquad \text{(V.19)} \\
= \;& \kappa\breve{u}w, \qquad \text{(V.20)}
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{3(11)} \approx \;& \hat{f}^a{}_1(\hat{f}_{13,a} - \gamma^b_{1a}\hat{f}_{b3} - \gamma^b_{3a}\hat{f}_{1b}) + \hat{f}_3{}^a(\hat{f}_{a1,1} - \gamma^b_{a1}\hat{f}_{b1} - \gamma^b_{11}\hat{f}_{ab}) \\
&+\frac{1}{8}(\,\ell_{,3}g_{11} - \,\ell_{,1}g_{13} - \,\ell_{,1}g_{13}) \\
&+\frac{2\pi}{c}\left(\hat{j}^4\hat{f}_{34}g_{11} + \frac{1}{3}\hat{j}^2\hat{f}_{21}g_{13} + \frac{1}{3}\hat{j}^2\hat{f}_{21}g_{13}\right) \qquad \text{(V.21)} \\
= \;& \breve{u}(-\gamma_{311}i\grave{u} + \gamma_{131}\breve{u}) - i\grave{u}(-\gamma_{131}\breve{u} + \gamma_{311}i\grave{u}) \qquad \text{(V.22)} \\
= \;& -\kappa w^2. \qquad \text{(V.23)}
\end{aligned}
$$

In tetrad form (2.63) becomes,

$$\Upsilon_{c\,[de]} \approx \frac{1}{2}(\hat{f}_{de,c} - \gamma^b_{dc}\hat{f}_{be} - \gamma^b_{ec}\hat{f}_{db} + \hat{f}_{ce,d} - \gamma^b_{cd}\hat{f}_{be} - \gamma^b_{ed}\hat{f}_{cb} - \hat{f}_{cd,e} + \gamma^b_{ce}\hat{f}_{bd} + \gamma^b_{de}\hat{f}_{cb}) + \frac{4\pi}{3c}\left(\hat{j}_d g_{ec} - \hat{j}_e g_{dc}\right), \tag{V.24}$$

$$\Upsilon_{1[12]} \approx \frac{1}{2}(\hat{f}_{12,1} - \gamma^b_{11}\hat{f}_{b2} - \gamma^b_{21}\hat{f}_{1b} + \hat{f}_{12,1} - \gamma^b_{11}\hat{f}_{b2} - \gamma^b_{21}\hat{f}_{1b} - \hat{f}_{11,2} + \gamma^b_{12}\hat{f}_{b1} + \gamma^b_{12}\hat{f}_{1b}) + \frac{4\pi}{3c}\left(\hat{j}_1 g_{21} - \hat{j}_2 g_{11}\right) \tag{V.25}$$

$$= -D\breve{u} + \frac{4\pi}{3c}\hat{j}^2, \tag{V.26}$$

$$\Upsilon_{1[23]} \approx \frac{1}{2}(\hat{f}_{23,1} - \gamma^b_{21}\hat{f}_{b3} - \gamma^b_{31}\hat{f}_{2b} + \hat{f}_{13,2} - \gamma^b_{12}\hat{f}_{b3} - \gamma^b_{32}\hat{f}_{1b} - \hat{f}_{12,3} + \gamma^b_{13}\hat{f}_{b2} + \gamma^b_{23}\hat{f}_{1b}) + \frac{4\pi}{3c}\left(\hat{j}_2 g_{31} - \hat{j}_3 g_{21}\right) \tag{V.27}$$

$$= \frac{1}{2}(\delta\breve{u} - \gamma_{321}i\grave{u} - \gamma_{231}\breve{u} - \gamma_{312}i\grave{u} + \gamma_{132}\breve{u}) - \frac{4\pi}{3c}\hat{j}_3 \tag{V.28}$$

$$= \frac{1}{2}(\delta\breve{u} - \pi^* w^* - \tau w) - \frac{4\pi}{3c}\hat{j}_3 \tag{V.29}$$

$$= \frac{1}{2}(\delta\breve{u} - i\delta\grave{u}) - \frac{2\pi}{3c}\hat{j}^4, \tag{V.30}$$

$$\Upsilon_{3[12]} \approx \frac{1}{2}(\hat{f}_{12,3} - \gamma^b_{13}\hat{f}_{b2} - \gamma^b_{23}\hat{f}_{1b} + \hat{f}_{32,1} - \gamma^b_{31}\hat{f}_{b2} - \gamma^b_{21}\hat{f}_{3b} - \hat{f}_{31,2} + \gamma^b_{32}\hat{f}_{b1} + \gamma^b_{12}\hat{f}_{3b}) + \frac{4\pi}{3c}\left(\hat{j}_1 g_{23} - \hat{j}_2 g_{13}\right) \tag{V.31}$$

$$= \frac{1}{2}(-\delta\breve{u} + \gamma_{231}\breve{u} + \gamma_{321}i\grave{u} + \gamma_{132}\breve{u} - \gamma_{312}i\grave{u}) \tag{V.32}$$

$$= -\frac{1}{2}(\delta\breve{u} + \tau w - \pi^* w^*), \tag{V.33}$$

$$\Upsilon_{1[13]} \approx \frac{1}{2}(\hat{f}_{13,1} - \gamma^b_{11}\hat{f}_{b3} - \gamma^b_{31}\hat{f}_{1b} + \hat{f}_{13,1} - \gamma^b_{11}\hat{f}_{b3} - \gamma^b_{31}\hat{f}_{1b} - \hat{f}_{11,3} + \gamma^b_{13}\hat{f}_{b1} + \gamma^b_{13}\hat{f}_{1b}) + \frac{4\pi}{3c}\left(\hat{j}_1 g_{31} - \hat{j}_3 g_{11}\right) \tag{V.34}$$

$$= -\gamma_{311}i\grave{u} + \gamma_{131}\breve{u} \tag{V.35}$$

$$= -\kappa w. \tag{V.36}$$

Appendix W

Kursunoglu's theory with sources and non-Abelian fields

Kursunoglu's theory[23] is roughly the electromagnetic dual of our theory, except it does not allow sources, and it does not ordinarily allow a non-zero total cosmological constant, and the Lagrangian density does not ordinarily use our non-symmetric Ricci tensor (2.5). Here we show that Kursunoglu's theory can be generalized to include sources, and it can also be generalized to non-Abelian fields, but in both cases it must be done in a rather ugly and inelegant way. We also allow for $\Lambda\neq 0$ and show that the theory can be derived from a Lagrangian density which contains the non-symmetric Ricci tensor (2.5). The Lagrangian density is

$$\begin{aligned}\mathcal{L}(\widehat\Gamma^\lambda_{\rho\tau},N_{\rho\tau}) &= -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}\mathcal{R}_{\nu\mu}(\widehat\Gamma)+(n-2)\Lambda_b\right]\\ &\quad-\frac{1}{16\pi}\sqrt{-g}\,(n-2)\Lambda_z+\mathcal{L}_m(u^\nu,\psi_e,g_{\mu\nu},A_\nu\ldots),\end{aligned} \tag{W.1}$$

where $\Lambda = \Lambda_b + \Lambda_z$ matches measurement, and the metric and electromagnetic field are defined to be

$$\sqrt{-g}\, g^{\mu\nu} = \sqrt{-N} N^{\dashv(\mu\nu)}, \qquad \sqrt{-g}\, F_{\alpha\rho} = \frac{1}{2\sqrt{2}\, i}\, \varepsilon_{\alpha\rho\mu\nu} \sqrt{-N} N^{\dashv[\nu\mu]} \Lambda_b^{1/2}. \tag{W.2}$$

It is helpful to decompose $\widehat{\Gamma}^\alpha_{\nu\mu}$ into a new connection $\tilde{\Gamma}^\alpha_{\nu\mu}$ and a vector B_μ,

$$\widehat{\Gamma}^\alpha_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu B_\nu - \delta^\alpha_\nu B_\mu)\sqrt{2}\, i \Lambda_b^{1/2}, \tag{W.3}$$

$$\text{where} \quad \tilde{\Gamma}^\alpha_{\nu\mu} = \widehat{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu \widehat{\Gamma}^\sigma_{[\sigma\nu]} - \delta^\alpha_\nu \widehat{\Gamma}^\sigma_{[\sigma\mu]})/(n-1), \tag{W.4}$$

$$B_\nu = \frac{1}{(n-1)\sqrt{-2\Lambda_b}}\, \widehat{\Gamma}^\sigma_{[\nu\sigma]}. \tag{W.5}$$

By contracting (W.4) on the right and left we see that $\tilde{\Gamma}^\alpha_{\nu\mu}$ has the symmetry

$$\tilde{\Gamma}^\alpha_{\nu\alpha} = \widehat{\Gamma}^\alpha_{(\nu\alpha)} = \tilde{\Gamma}^\alpha_{\alpha\nu}, \tag{W.6}$$

so it has only n^3–n independent components. Using $\mathcal{R}_{\nu\mu}(\widehat{\Gamma}) = \mathcal{R}_{\nu\mu}(\tilde{\Gamma}) + 2B_{[\nu,\mu]}\sqrt{2}\, i \Lambda_b^{1/2}$ from (R.17), the Lagrangian density (W.1) can be rewritten in terms of $\tilde{\Gamma}^\alpha_{\nu\mu}$ and B_σ,

$$\begin{aligned} \mathcal{L}(\widehat{\Gamma}^\lambda_{\rho\tau}, N_{\rho\tau}) = & -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}(\tilde{\mathcal{R}}_{\nu\mu} + 2B_{[\nu,\mu]}\sqrt{2}\, i\Lambda_b^{1/2}) + (n-2)\Lambda_b\right] \\ & -\frac{1}{16\pi}\sqrt{-g}\,(n-2)\Lambda_z + \mathcal{L}_m(u^\nu, \psi_e, g_{\mu\nu}, A_\sigma \ldots). \end{aligned} \tag{W.7}$$

Setting $\delta\mathcal{L}/\delta B_\mu = 0$ and using (W.2) gives Faraday's law,

$$(\sqrt{-g}\, \varepsilon^{\tau\omega\alpha\rho} F_{\alpha\rho})_{,\omega} = 0. \tag{W.8}$$

Using $\sqrt{-g}\, \varepsilon^{\tau\omega\alpha\rho} = \epsilon^{\tau\omega\alpha\rho}$, this equation is satisfied if we let

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{W.9}$$

Therefore, we can alter the Lagrangian by using (W.2) to define $\sqrt{-N} N^{\dashv[\mu\nu]}$ as

$$\sqrt{-N} N^{\dashv[\mu\nu]} = \sqrt{-g}\, \varepsilon^{\nu\mu\alpha\rho} F_{\alpha\rho} \Lambda_b^{-1/2}/\sqrt{2}\, i. \tag{W.10}$$

With this definition the term

$$\sqrt{-g}\,\varepsilon^{\mu\nu\alpha\rho}A_{[\mu,\nu]}B_{[\alpha,\rho]} = (\sqrt{-g}\,\varepsilon^{\mu\nu\alpha\rho}A_{\mu,\nu}B_{\alpha})_{,\rho} \tag{W.11}$$

is a total divergence so it can be removed from the Lagrangian density.

Therefore we can rewrite the Lagrangian density again as

$$\begin{aligned}\mathcal{L}(\tilde\Gamma^\lambda_{\rho\tau}, g_{\rho\tau}, A_\mu) \;&=\; -\frac{1}{16\pi}\sqrt{-N}\left[N^{\dashv\mu\nu}\tilde{\mathcal{R}}_{\nu\mu} + (n-2)\Lambda_b\right]\\ &\quad -\frac{1}{16\pi}\sqrt{-g}\,(n-2)\Lambda_z + \mathcal{L}_m(u^\nu, \psi_e, g_{\mu\nu}, A_\sigma\dots),\end{aligned} \tag{W.12}$$

where $N_{\mu\nu}$ is defined as

$$\sqrt{-N}N^{\dashv\mu\nu} = \sqrt{-g}\,g^{\mu\nu} + \sqrt{-g}\,\varepsilon^{\nu\mu\alpha\rho}F_{\alpha\rho}\Lambda_b^{-1/2}/\sqrt{2}\,i. \tag{W.13}$$

Setting $\delta\mathcal{L}/\delta(\sqrt{-g}g^{\mu\nu}) = 0$ gives the same Einstein equations as before

$$\tilde{\mathcal{R}}_{(\nu\mu)} + \Lambda_b N_{(\nu\mu)} + \Lambda_z g_{\nu\mu} = 8\pi\left(T_{\nu\mu} - \frac{1}{(n-2)}g_{\nu\mu}T^\alpha_\alpha\right), \tag{W.14}$$

$$\tilde{G}_{\nu\mu} = 8\pi T_{\nu\mu} - \Lambda_b\left(N_{(\nu\mu)} - \frac{1}{2}g_{\nu\mu}N^\rho_\rho\right) + \Lambda_z\left(\frac{n}{2}-1\right)g_{\nu\mu}, \tag{W.15}$$

and setting $\delta\mathcal{L}/\delta\tilde\Gamma^\alpha_{\nu\mu} = 0$ gives the same connection equations but without a j^ν term,

$$N_{\nu\mu,\beta} - \tilde\Gamma^\alpha_{\nu\beta}N_{\alpha\mu} - \tilde\Gamma^\alpha_{\beta\mu}N_{\nu\alpha} = 0. \tag{W.16}$$

Setting $\delta\mathcal{L}/\delta A_\nu = 0$ and using $\varepsilon^{\nu\mu\omega\tau}\varepsilon_{\nu\mu\omega\lambda} = -6\delta^\tau_\lambda$ gives Kursunoglu's version of Ampere's law

$$0 \;=\; \frac{4\pi}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}}{\partial A_\tau} - \left(\frac{\partial\mathcal{L}}{\partial A_{\tau,\omega}}\right)_{,\omega}\right] \tag{W.17}$$

$$= \frac{4\pi}{\sqrt{-g}}\left[\frac{1}{16\pi}\left(\frac{\partial(\sqrt{-N}N^{\dashv[\mu\nu]})}{\partial A_{\tau,\omega}}\left(\tilde{\mathcal{R}}_{\nu\mu} + (n-2)\Lambda_b\frac{\partial\sqrt{-N}}{\partial(\sqrt{-N}N^{\dashv\mu\nu})}\right)\right)_{,\omega}\right] - 4\pi j^\tau \tag{W.18}$$

$$= \frac{\Lambda_b^{-1/2}}{2\sqrt{2}\,i\sqrt{-g}}\left(\sqrt{-g}\,\varepsilon^{\nu\mu\alpha\rho}\delta^\tau_\rho\delta^\omega_\alpha(\tilde{\mathcal{R}}_{\nu\mu} + \Lambda_b N_{\nu\mu})\right)_{,\omega} - 4\pi j^\tau \tag{W.19}$$

$$= \frac{\Lambda_b^{1/2}}{2\sqrt{2}\,i}(\varepsilon^{\nu\mu\omega\tau}(N_{[\nu\mu]} + \tilde{\mathcal{R}}_{[\nu\mu]}/\Lambda_b))_{;\omega} - 4\pi j^\tau \quad \text{(W.20)}$$

$$= \frac{\Lambda_b^{1/2}}{2\sqrt{2}\,i}\varepsilon^{\nu\mu\omega\tau}(N_{[\nu\mu,\omega]} + \tilde{\mathcal{R}}_{[\nu\mu,\omega]}/\Lambda_b) - 4\pi j^\tau \quad \text{(W.21)}$$

where j^τ is defined the same as before

$$j^\tau = \frac{-1}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}_m}{\partial A_\tau} - \left(\frac{\partial\mathcal{L}_m}{\partial A_{\tau,\omega}}\right)_{,\omega}\right]. \quad \text{(W.22)}$$

This is satisfactory because from (W.2) we have

$$\frac{\Lambda_b^{1/2}}{2\sqrt{2}\,i}\varepsilon^{\nu\mu\omega\tau}N_{[\nu\mu]} = \frac{\Lambda_b^{1/2}}{2\sqrt{2}\,i}\varepsilon_{\nu\mu}{}^{\omega\tau}\left(\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv[\mu\nu]} + \mathcal{O}(\Lambda_b^{-3/2})\right) = F^{\omega\tau} + \mathcal{O}(\Lambda_b^{-1}). \quad \text{(W.23)}$$

The identities (4.3,4.4) without the j^ν terms become

$$\tilde{G}^\sigma_{\nu;\sigma} = \frac{3}{2}\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv[\rho\sigma]}\,\tilde{\mathcal{R}}_{[\sigma\rho,\nu]}, \quad \text{(W.24)}$$

$$\left(N^{(\mu}{}_{\nu)} - \frac{1}{2}\delta^\mu_\nu N^\rho_\rho\right)_{;\mu} = \frac{3}{2}\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv[\rho\sigma]}N_{[\sigma\rho,\nu]}. \quad \text{(W.25)}$$

The ordinary Lorentz force equation results from taking the divergence of the Einstein equations (W.15) using (W.24,W.25,W.21,W.2)

$$8\pi T^\sigma_{\nu;\sigma} = \tilde{G}^\sigma_{\nu;\sigma} + \Lambda_b\left(N^{(\mu}{}_{\nu)} - \frac{1}{2}\delta^\mu_\nu N^\rho_\rho\right)_{;\mu} \quad \text{(W.26)}$$

$$= \frac{3}{2}\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv[\rho\sigma]}(\tilde{\mathcal{R}}_{[\sigma\rho,\nu]} + \Lambda_b N_{[\sigma\rho,\nu]}) \quad \text{(W.27)}$$

$$= \frac{3}{2}\frac{\sqrt{-N}}{\sqrt{-g}}N^{\dashv[\sigma\rho]}\varepsilon_{\sigma\rho\nu\tau}\frac{8\pi\sqrt{2}\,i\Lambda_b^{1/2}}{6}j^\tau \quad \text{(W.28)}$$

$$= \frac{3}{2}F_{\tau\nu}\frac{2\sqrt{2}\,i}{\Lambda_b^{1/2}}\frac{8\pi\sqrt{2}\,i\Lambda_b^{1/2}}{6}j^\tau \quad \text{(W.29)}$$

$$T^\sigma_{\nu;\sigma} = F_{\nu\sigma}j^\sigma. \quad \text{(W.30)}$$

Let us see if Kursunoglu's theory works for the non-Abelian case. We define the electro-weak field tensor $f^{\nu\mu}$ as

$$\mathsf{g}^{1/2d}f_{\alpha\rho} = \frac{1}{2\sqrt{2}i}N^{1/2d}\varepsilon_{\alpha\rho\mu\nu}N^{\dashv[\nu\mu]}\Lambda_b^{1/2}. \quad \text{(W.31)}$$

We will use $\mathcal{A}_\tau$ as in the non-Abelian Lagrangian density (7.20) instead of B_τ as in (W.7). Setting $\delta\mathcal{L}/\delta\mathcal{A}_\tau = 0$ gives the dual of (7.47), which is the Weinberg-Salam equivalent of Faraday's law,

$$(\mathsf{g}^{1/2d}\varepsilon^{\tau\omega\alpha\rho}f_{\alpha\rho})_{,\omega} - \sqrt{-2\Lambda_b}\,\mathsf{g}^{1/2d}\varepsilon^{\tau\omega\alpha\rho}[f_{\alpha\rho}, \mathcal{A}_\omega] = 0. \tag{W.32}$$

Using $\mathsf{g}^{1/2d}\varepsilon^{\tau\omega\alpha\rho} = \epsilon^{\tau\omega\alpha\rho}$, this equation is satisfied if we let

$$f_{\alpha\rho} = 2\mathcal{A}_{[\rho,\alpha]} + \sqrt{-2\Lambda_b}[\mathcal{A}_\alpha, \mathcal{A}_\rho], \tag{W.33}$$

as seen by substituting (W.33) into (W.32),

$$\begin{aligned} 0 &= \epsilon^{\tau\omega\alpha\rho}(2\mathcal{A}_{[\rho,\alpha],\omega} + \sqrt{-2\Lambda_b}([\mathcal{A}_\alpha, \mathcal{A}_\rho]_{,\omega} - [2\mathcal{A}_{[\rho,\alpha]}, \mathcal{A}_\omega]) \\ &\qquad + 2\Lambda_b[[\mathcal{A}_\alpha, \mathcal{A}_\rho], \mathcal{A}_\omega]) &\text{(W.34)} \\ &= \epsilon^{\tau\omega\alpha\rho}(2\mathcal{A}_{\rho,\alpha,\omega} + 2\sqrt{-2\Lambda_b}(\mathcal{A}_{\alpha,\omega}\mathcal{A}_\rho + \mathcal{A}_\alpha\mathcal{A}_{\rho,\omega} - \mathcal{A}_{\rho,\alpha}\mathcal{A}_\omega + \mathcal{A}_\omega\mathcal{A}_{\rho,\alpha}) \\ &\qquad + 4\Lambda_b(\mathcal{A}_\alpha\mathcal{A}_\rho\mathcal{A}_\omega - \mathcal{A}_\omega\mathcal{A}_\alpha\mathcal{A}_\rho)) &\text{(W.35)} \\ &= 0. &\text{(W.36)} \end{aligned}$$

Therefore we can alter the Lagrangian by using (W.31,W.33) and $\varepsilon_{\alpha\rho\mu\nu}\varepsilon^{\mu\nu\tau\sigma} = -4\delta^\tau_{[\alpha}\delta^\sigma_{\rho]}$ to define $N^{1/2d}N^{\dashv[\nu\mu]}$ as

$$N^{1/2d}N^{\dashv[\mu\nu]} = \mathsf{g}^{1/2d}\varepsilon^{\nu\mu\alpha\rho}f_{\alpha\rho}\Lambda_b^{-1/2}/\sqrt{2}\,i. \tag{W.37}$$

With this definition we have a term $tr(g^{1/2d}\varepsilon^{\alpha\rho\nu\mu}f_{\alpha\rho}f_{\nu\mu})$ in the Lagrangian density. Expanding this out we find that the terms are all total divergences or zero,

$$\begin{aligned} tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_{[\rho,\alpha]}\mathcal{A}_{[\mu,\nu]}) &= tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_{\rho,\alpha}\mathcal{A}_{\mu,\nu}) \\ &= tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_{\rho,\alpha}\mathcal{A}_\mu)_{,\nu}, &\text{(W.38)} \end{aligned}$$

$$
\begin{aligned}
tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_{[\rho,\alpha]}[\mathcal{A}_\nu,\mathcal{A}_\mu]) &= 2tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_{\rho,\alpha}\mathcal{A}_\nu\mathcal{A}_\mu)\\
&= 2tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_{\rho,\alpha}\mathcal{A}_\nu\mathcal{A}_\mu+\mathcal{A}_\rho\mathcal{A}_{\nu,\alpha}\mathcal{A}_\mu+\mathcal{A}_\rho\mathcal{A}_\nu\mathcal{A}_{\mu,\alpha})/3\\
&= 2tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_\rho\mathcal{A}_\nu\mathcal{A}_\mu)_{,\alpha}/3, \qquad \text{(W.39)}\\
tr(\varepsilon^{\alpha\rho\nu\mu}[\mathcal{A}_\alpha,\mathcal{A}_\rho][\mathcal{A}_\nu,\mathcal{A}_\mu]) &= 4tr(\varepsilon^{\alpha\rho\nu\mu}\mathcal{A}_\alpha\mathcal{A}_\rho\mathcal{A}_\nu\mathcal{A}_\mu)\\
&= -4tr(\varepsilon^{\mu\alpha\rho\nu}\mathcal{A}_\alpha\mathcal{A}_\rho\mathcal{A}_\nu\mathcal{A}_\mu)\\
&= -4tr(\varepsilon^{\mu\alpha\rho\nu}\mathcal{A}_\mu\mathcal{A}_\alpha\mathcal{A}_\rho\mathcal{A}_\nu)=0. \qquad \text{(W.40)}
\end{aligned}
$$

So the term $tr(\mathsf{g}^{1/2d}\varepsilon^{\alpha\rho\nu\mu}f_{\alpha\rho}f_{\nu\mu})$ can be removed from the Lagrangian density. Therefore we can write the Lagrangian density in a form similar to (W.12),

$$
\begin{aligned}
\mathcal{L}(\tilde\Gamma^\lambda_{\rho\tau},\mathsf{g}_{\rho\tau},\mathcal{A}_\mu) &= -\frac{1}{16\pi}N^{1/2d}\,tr\Big[N^{\dashv\mu\nu}\tilde{\mathcal{R}}_{\nu\mu}+(n-2)\Lambda_b\Big]\\
&\quad -\frac{1}{16\pi}\mathsf{g}^{1/2d}(n-2)\Lambda_z+\mathcal{L}_m(u^\nu,\psi_e,g_{\mu\nu},\mathcal{A}_\sigma\dots), \qquad \text{(W.41)}
\end{aligned}
$$

where $N_{\mu\nu}$ is defined as

$$
N^{1/2d}N^{\dashv\mu\nu}=\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}+\mathsf{g}^{1/2d}\varepsilon^{\nu\mu\alpha\rho}f_{\alpha\rho}\Lambda_b^{-1/2}/\sqrt{2}\,i. \qquad \text{(W.42)}
$$

The calculations subsequent to (W.12) are also similar for the non-Abelian case. So the non-Abelian Kursunoglu's theory works in a way. However it is a bit questionable because we must assume no $\mathcal{L}_m$ when calculating (W.32) and its solution (W.33), and then we subsequently introduce a $\mathcal{L}_m(\mathcal{A}_\nu)$ term after the sourceless Lagrangian density is derived. This problem does not occur for the Abelian case because the A_ν in (W.9) is independent of the B_ν in (W.7). The problem also does not occur with the theory considered in §7, where we use the direct definition of the elector-weak field instead of the dual definition.

Appendix X

Possible extension of the theory to non-Abelian symmetric fields

Now let us calculate the field equations with the Lagrangian density (7.20) and the special case $\mathcal{A}_\nu = 0$, $N^{\dashv[\mu\nu]} = 0$. The theory we get depends on how we define the physical metric $g^{\nu\mu}$, and there are several possible definitions (7.11) which are consistent with the Abelian case (2.4). We will use what appears to be the simplest definition

$$\mathsf{g}^{1/2d}\mathsf{g}^{\nu\mu} = N^{1/2d} N^{\dashv(\nu\mu)}, \tag{X.1}$$

$$g^{1/2d} g^{\nu\mu} = I tr[\mathsf{g}^{1/2d}\mathsf{g}^{\nu\mu}]/d. \tag{X.2}$$

The theory we get also depends on whether we use $\Lambda_z\sqrt{-g}$ or $\Lambda_z \mathsf{g}^{1/2d}$ in the Lagrangian density (7.20). If we use $\Lambda_z \mathsf{g}^{1/2d}$ we get $\Lambda_z \mathsf{g}^{1/2d} + \Lambda_b N^{1/2d} = \Lambda \mathsf{g}^{1/2d}$ for symmetric fields, so that the cosmological constant terms have essentially no effect on the theory. If we instead use $\Lambda_z\sqrt{-g}$, the difference between this term and $\Lambda_b N^{1/2d}$ has the effect of giving a mass to the field associated with the traceless part of $\mathsf{g}_{\mu\nu}$. In fact we will find that this mass term is imaginary $m=i\hbar\sqrt{2\Lambda_b}$, so unless something unusual is happening, the use of $\Lambda_z\sqrt{-g}$ with the metric (X.2) is probably not going to give a physically acceptable theory. Nevertheless, in the following calculations we will use $\Lambda_z\sqrt{-g}$ in the Lagrangian density (7.20) because it is effectively the more general case. It happens that the results for the choice $\Lambda_z \mathsf{g}^{1/2d}$ can be obtained by setting to zero the mass term that occurs in the theory, which is any term with a Λ_b factor.

Setting $\delta\mathcal{L}/\delta(N^{1/2d}N^{\dashv(\mu\nu)})=0$ using the identities $N=[det(N^{1/2d}N^{\dashv\mu\nu})]^{2/(n-2)}$ and $\mathsf{g}=[det(N^{1/2d}N^{\dashv(\mu\nu)})]^{2/(n-2)}$ gives our equivalent of the Einstein equations,

$$\tilde{\mathcal{R}}_{(\nu\mu)}+\Lambda_b\mathsf{g}_{\nu\mu}+\Lambda_z g_{\nu\mu}=8\pi S_{\nu\mu} \tag{X.3}$$

$$\text{where}\quad S_{\nu\mu}\equiv 2\frac{\delta\mathcal{L}_m}{\delta(N^{1/2d}N^{(\mu\nu)})}=2\frac{\delta\mathcal{L}_m}{\delta(\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu})}\,. \tag{X.4}$$

For the present let us assume that $S_{\nu\mu}=0$. Using

$$\tilde{\mathcal{R}}_{\nu\mu} \;=\; \tilde\Gamma^\alpha_{\nu\mu,\alpha}-\tilde\Gamma^\alpha_{\alpha(\nu,\mu)}+\frac{1}{2}\tilde\Gamma^\sigma_{\nu\mu}\tilde\Gamma^\alpha_{\sigma\alpha}+\frac{1}{2}\tilde\Gamma^\alpha_{\sigma\alpha}\tilde\Gamma^\sigma_{\nu\mu}-\tilde\Gamma^\sigma_{\nu\alpha}\tilde\Gamma^\alpha_{\sigma\mu}, \tag{X.5}$$

and the definition

$$\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}=\frac{\partial\mathcal{L}}{\partial\tilde\Gamma^\beta_{\tau\rho}}-\left(\frac{\partial\mathcal{L}}{\partial\tilde\Gamma^\beta_{\tau\rho,\omega}}\right)_{,\omega}\cdots \tag{X.6}$$

we can calculate

$$\begin{aligned}
-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}} \;=\;& \frac{1}{2}\delta^\sigma_\beta\delta^\tau_\nu\delta^\rho_\mu\tilde\Gamma^\alpha_{\sigma\alpha}N^{1/2d}N^{\dashv\mu\nu}+\frac{1}{2}N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\sigma_{\nu\mu}\delta^\alpha_\beta\delta^\tau_\sigma\delta^\rho_\alpha\\
&+\frac{1}{2}N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\alpha_{\sigma\alpha}\delta^\sigma_\beta\delta^\tau_\nu\delta^\rho_\mu+\frac{1}{2}\delta^\alpha_\beta\delta^\tau_\sigma\delta^\rho_\alpha\tilde\Gamma^\sigma_{\nu\mu}N^{1/2d}N^{\dashv\mu\nu}\\
&-\delta^\sigma_\beta\delta^\tau_\nu\delta^\rho_\alpha\tilde\Gamma^\alpha_{\sigma\mu}N^{1/2d}N^{\dashv\mu\nu}-N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\sigma_{\nu\alpha}\delta^\alpha_\beta\delta^\tau_\sigma\delta^\rho_\mu\\
&-2(N^{1/2d}N^{\dashv\mu\nu}\delta^\alpha_\beta\delta^\tau_\nu\delta^\rho_{[\mu}\delta^\omega_{\alpha]})_{,\omega}-(N^{1/2d}N^{\dashv\mu\nu}\delta^\alpha_\beta\delta^\tau_\alpha\delta^\rho_{[\nu}\delta^\omega_{\mu]})_{,\omega}\\
=\;& -(N^{1/2d}N^{\dashv\rho\tau})_{,\beta}-\tilde\Gamma^\rho_{\beta\mu}N^{1/2d}N^{\dashv\mu\tau}-N^{1/2d}N^{\dashv\rho\nu}\tilde\Gamma^\tau_{\nu\beta}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\tilde\Gamma^\alpha_{\beta\alpha}N^{1/2d}N^{\dashv\rho\tau}+\frac{1}{2}N^{1/2d}N^{\dashv\rho\tau}\tilde\Gamma^\alpha_{\beta\alpha}\\
&+\delta^\rho_\beta\left((N^{1/2d}N^{\dashv\omega\tau})_{,\omega}+\frac{1}{2}N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\tau_{\nu\mu}+\frac{1}{2}\tilde\Gamma^\tau_{\nu\mu}N^{1/2d}N^{\dashv\mu\nu}\right)\\
&+\delta^\tau_\beta(N^{1/2d}N^{\dashv[\rho\omega]})_{,\omega}, \qquad \text{(X.7)}
\end{aligned}
$$

$$
\begin{aligned}
-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\alpha\rho}} &= (n{-}2)(N^{1/2d}N^{\dashv[\rho\omega]})_{,\omega}+\frac{1}{2}\left(N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\rho_{\nu\mu}-\tilde\Gamma^\rho_{\nu\mu}N^{1/2d}N^{\dashv\mu\nu}\right)\\
&+\frac{1}{2}\left(\tilde\Gamma^\alpha_{\tau\alpha}N^{1/2d}N^{\dashv\rho\tau}-N^{1/2d}N^{\dashv\rho\tau}\tilde\Gamma^\alpha_{\tau\alpha}\right), \qquad \text{(X.8)}\\
-16\pi\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\tau\alpha}} &= (n{-}1)\left((N^{1/2d}N^{\dashv\omega\tau})_{,\omega}+\frac{1}{2}N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\tau_{\nu\mu}+\frac{1}{2}\tilde\Gamma^\tau_{\nu\mu}N^{1/2d}N^{\dashv\mu\nu}\right)\\
&+(N^{1/2d}N^{\dashv[\tau\omega]})_{,\omega}-\frac{1}{2}\left(N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\tau_{\nu\mu}-\tilde\Gamma^\tau_{\nu\mu}N^{1/2d}N^{\dashv\mu\nu}\right)\\
&-\frac{1}{2}\left(\tilde\Gamma^\alpha_{\rho\alpha}N^{1/2d}N^{\dashv\rho\tau}-N^{1/2d}N^{\dashv\rho\tau}\tilde\Gamma^\alpha_{\rho\alpha}\right). \qquad \text{(X.9)}
\end{aligned}
$$

Setting $\delta\mathcal{L}/\delta\tilde\Gamma^\beta_{\tau\rho}{=}0$ using a Lagrange multiplier $tr[\Omega^\nu\tilde\Gamma^\alpha_{[\alpha\nu]}]$ to enforce (7.18) gives

$$
\begin{aligned}
0 &= 16\pi\left[\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\beta_{\tau\rho}}-\frac{\delta^\tau_\beta}{(n{-}1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\alpha\rho}}-\frac{\delta^\rho_\beta}{(n{-}1)}\frac{\Delta\mathcal{L}}{\Delta\tilde\Gamma^\alpha_{\tau\alpha}}\right]\\
&= (N^{1/2d}N^{\dashv\rho\tau})_{,\beta}+\tilde\Gamma^\rho_{\beta\mu}N^{1/2d}N^{\dashv\mu\tau}+N^{1/2d}N^{\dashv\rho\nu}\tilde\Gamma^\tau_{\nu\beta}\\
&-\frac{1}{2}\tilde\Gamma^\alpha_{\beta\alpha}N^{1/2d}N^{\dashv\rho\tau}-\frac{1}{2}N^{1/2d}N^{\dashv\rho\tau}\tilde\Gamma^\alpha_{\beta\alpha}\\
&+\frac{\delta^\tau_\beta}{2(n{-}1)}\left(N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\rho_{\nu\mu}-\tilde\Gamma^\rho_{\nu\mu}N^{1/2d}N^{\dashv\mu\nu}+\tilde\Gamma^\alpha_{\sigma\alpha}N^{1/2d}N^{\dashv\sigma\rho}-N^{1/2d}N^{\dashv\sigma\rho}\tilde\Gamma^\alpha_{\sigma\alpha}\right)\\
&-\frac{\delta^\rho_\beta}{2(n{-}1)}\left(N^{1/2d}N^{\dashv\mu\nu}\tilde\Gamma^\tau_{\nu\mu}-\tilde\Gamma^\tau_{\nu\mu}N^{1/2d}N^{\dashv\mu\nu}+\tilde\Gamma^\alpha_{\sigma\alpha}N^{1/2d}N^{\dashv\sigma\tau}-N^{1/2d}N^{\dashv\sigma\tau}\tilde\Gamma^\alpha_{\sigma\alpha}\right)\\
&+\frac{1}{(n{-}1)}(\delta^\rho_\beta(N^{1/2d}N^{\dashv[\tau\omega]})_{,\omega}-\delta^\tau_\beta(N^{1/2d}N^{\dashv[\rho\omega]})_{,\omega}). \qquad \text{(X.10)}
\end{aligned}
$$

Using $N^{\dashv[\mu\nu]}{=}0$ and the metric definition (X.2) gives

$$
\begin{aligned}
0 &= (\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau})_{,\beta}+\tilde\Gamma^\rho_{\beta\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\tau}+\mathsf{g}^{1/2d}\mathsf{g}^{\rho\nu}\tilde\Gamma^\tau_{\nu\beta}-\frac{1}{2}\tilde\Gamma^\alpha_{\beta\alpha}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau}-\frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau}\tilde\Gamma^\alpha_{\beta\alpha}\\
&+\frac{\delta^\tau_\beta}{2(n{-}1)}\left(\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\tilde\Gamma^\rho_{\nu\mu}-\tilde\Gamma^\rho_{\nu\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}+\tilde\Gamma^\alpha_{\sigma\alpha}\mathsf{g}^{1/2d}\mathsf{g}^{\sigma\rho}-\mathsf{g}^{1/2d}\mathsf{g}^{\sigma\rho}\tilde\Gamma^\alpha_{\sigma\alpha}\right)\\
&-\frac{\delta^\rho_\beta}{2(n{-}1)}\left(\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\tilde\Gamma^\tau_{\nu\mu}-\tilde\Gamma^\tau_{\nu\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}+\tilde\Gamma^\alpha_{\sigma\alpha}\mathsf{g}^{1/2d}\mathsf{g}^{\sigma\tau}-\mathsf{g}^{1/2d}\mathsf{g}^{\sigma\tau}\tilde\Gamma^\alpha_{\sigma\alpha}\right). \qquad \text{(X.11)}
\end{aligned}
$$

Contracting with $\mathsf{g}_{\tau\rho}$ and taking the trace gives

$$0 = tr[-(\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau})_{,\beta}\mathsf{g}_{\tau\rho}] + (n-2)tr[\tilde{\Gamma}^{\rho}_{\beta\rho}]\mathsf{g}^{1/2d} \tag{X.12}$$

$$= -dn(\mathsf{g}^{1/2d})_{,\beta} - \mathsf{g}^{1/2d}tr[\mathsf{g}^{\rho\tau}{}_{,\beta}\mathsf{g}_{\tau\rho}] + (n-2)tr[\tilde{\Gamma}^{\rho}_{\beta\rho}]\mathsf{g}^{1/2d} \tag{X.13}$$

$$= (n-2)(-d(\mathsf{g}^{1/2d})_{,\beta} + \mathsf{g}^{1/2d}tr[\tilde{\Gamma}^{\rho}_{\beta\rho}]). \tag{X.14}$$

Rewriting again gives

$$\begin{aligned} 0 = {} & \mathsf{g}^{\rho\tau}{}_{,\beta} + \tilde{\Gamma}^{\rho}_{\beta\mu}\mathsf{g}^{\mu\tau} + \mathsf{g}^{\rho\nu}\tilde{\Gamma}^{\tau}_{\nu\beta} - \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\mathsf{g}^{\rho\tau} - \frac{1}{2}\mathsf{g}^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\beta\alpha} + \mathsf{g}^{\rho\tau}tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d \\ & + \frac{\delta^{\tau}_{\beta}}{2(n-1)}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\rho}_{\nu\mu} - \tilde{\Gamma}^{\rho}_{\nu\mu}\mathsf{g}^{\mu\nu} + \tilde{\Gamma}^{\alpha}_{\sigma\alpha}\mathsf{g}^{\sigma\rho} - \mathsf{g}^{\sigma\rho}\tilde{\Gamma}^{\alpha}_{\sigma\alpha}\right) \\ & - \frac{\delta^{\rho}_{\beta}}{2(n-1)}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} - \tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{\mu\nu} + \tilde{\Gamma}^{\alpha}_{\sigma\alpha}\mathsf{g}^{\sigma\tau} - \mathsf{g}^{\sigma\tau}\tilde{\Gamma}^{\alpha}_{\sigma\alpha}\right). \end{aligned} \tag{X.15}$$

Multiplying on the left by $\mathsf{g}_{\omega\rho}$ and on the right by $\mathsf{g}_{\tau\lambda}$ gives

$$\begin{aligned} 0 = {} & -\mathsf{g}_{\omega\lambda,\beta} + \mathsf{g}_{\omega\rho}\tilde{\Gamma}^{\rho}_{\beta\lambda} + \tilde{\Gamma}^{\tau}_{\omega\beta}\mathsf{g}_{\tau\lambda} - \frac{1}{2}\mathsf{g}_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} - \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\mathsf{g}_{\omega\lambda} + \mathsf{g}_{\omega\lambda}tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d \\ & + \frac{\mathsf{g}_{\omega\rho}}{2(n-1)}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\rho}_{\nu\mu} - \tilde{\Gamma}^{\rho}_{\nu\mu}\mathsf{g}^{\mu\nu} + \tilde{\Gamma}^{\alpha}_{\sigma\alpha}\mathsf{g}^{\sigma\rho} - \mathsf{g}^{\sigma\rho}\tilde{\Gamma}^{\alpha}_{\sigma\alpha}\right)\mathsf{g}_{\beta\lambda} \\ & - \frac{\mathsf{g}_{\omega\beta}}{2(n-1)}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} - \tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{\mu\nu} + \tilde{\Gamma}^{\alpha}_{\sigma\alpha}\mathsf{g}^{\sigma\tau} - \mathsf{g}^{\sigma\tau}\tilde{\Gamma}^{\alpha}_{\sigma\alpha}\right)\mathsf{g}_{\tau\lambda}. \end{aligned} \tag{X.16}$$

Contracting gives

$$\begin{aligned} 0 = {} & \mathsf{g}^{\rho\tau}{}_{,\rho} + \tilde{\Gamma}^{\rho}_{\rho\mu}\mathsf{g}^{\mu\tau} + \mathsf{g}^{\rho\nu}\tilde{\Gamma}^{\tau}_{\nu\rho} - \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\rho\alpha}\mathsf{g}^{\rho\tau} - \frac{1}{2}\mathsf{g}^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\rho\alpha} + \mathsf{g}^{\rho\tau}tr[\tilde{\Gamma}^{\alpha}_{\rho\alpha}]/d \\ & + \frac{1}{2(n-1)}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} - \tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{\mu\nu} + \tilde{\Gamma}^{\alpha}_{\sigma\alpha}\mathsf{g}^{\sigma\tau} - \mathsf{g}^{\sigma\tau}\tilde{\Gamma}^{\alpha}_{\sigma\alpha}\right) \\ & - \frac{n}{2(n-1)}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} - \tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{\mu\nu} + \tilde{\Gamma}^{\alpha}_{\sigma\alpha}\mathsf{g}^{\sigma\tau} - \mathsf{g}^{\sigma\tau}\tilde{\Gamma}^{\alpha}_{\sigma\alpha}\right) \end{aligned} \tag{X.17}$$

$$\begin{aligned} = {} & \mathsf{g}^{\rho\tau}{}_{,\rho} + \tilde{\Gamma}^{\rho}_{\rho\mu}\mathsf{g}^{\mu\tau} + \mathsf{g}^{\rho\nu}\tilde{\Gamma}^{\tau}_{\nu\rho} - \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\rho\alpha}\mathsf{g}^{\rho\tau} - \frac{1}{2}\mathsf{g}^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\rho\alpha} + \mathsf{g}^{\rho\tau}tr[\tilde{\Gamma}^{\alpha}_{\rho\alpha}]/d \\ & - \frac{1}{2}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} - \tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{\mu\nu} + \tilde{\Gamma}^{\alpha}_{\sigma\alpha}\mathsf{g}^{\sigma\tau} - \mathsf{g}^{\sigma\tau}\tilde{\Gamma}^{\alpha}_{\sigma\alpha}\right) \end{aligned} \tag{X.18}$$

$$= \mathsf{g}^{\rho\tau}{}_{,\rho} + \mathsf{g}^{\rho\tau}tr[\tilde{\Gamma}^{\alpha}_{\rho\alpha}]/d + \frac{1}{2}\left(\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} + \tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{\mu\nu}\right) \tag{X.19}$$

Let us consider the possibility where only the traceless part of $\tilde{\Gamma}^\tau_{\nu\mu}$ is contained in $\tilde{\Gamma}^\tau_{[\nu\mu]}$, in which case (X.16) reduces to

$$0 \;=\; -\mathsf{g}_{\omega\lambda,\beta} + \mathsf{g}_{\omega\rho}\tilde{\Gamma}^\rho_{\beta\lambda} + \tilde{\Gamma}^\tau_{\omega\beta}\mathsf{g}_{\tau\lambda}. \tag{X.20}$$

From the trace of the symmetric part of this we see that $\tilde{\Gamma}^\tau_{(\nu\mu)}$ is given by the Christoffel connection composed from $tr(\mathsf{g}_{\nu\mu})$. From the antisymmetric part we see that $\tilde{\Gamma}^\tau_{[\nu\mu]}$ satisfies the equation

$$0 \;=\; \mathsf{g}_{\omega\rho}\tilde{\Gamma}^\rho_{[\beta\lambda]} + \tilde{\Gamma}^\tau_{[\omega\beta]}\mathsf{g}_{\tau\lambda} - \mathsf{g}_{\lambda\rho}\tilde{\Gamma}^\rho_{[\beta\omega]} - \tilde{\Gamma}^\tau_{[\lambda\beta]}\mathsf{g}_{\tau\omega} \tag{X.21}$$

Combining this with its permutations gives

$$0 \;=\; \mathsf{g}_{\lambda\rho}\tilde{\Gamma}^\rho_{[\beta\omega]} + \tilde{\Gamma}^\rho_{[\beta\omega]}\mathsf{g}_{\lambda\rho} \;\;\Rightarrow\;\; 0 = \tilde{\Gamma}^\nu_{[\beta\omega]} + \mathsf{g}^{\lambda\nu}\tilde{\Gamma}^\rho_{[\beta\omega]}\mathsf{g}_{\lambda\rho} \tag{X.22}$$

From this we find that

$$tr[\tilde{\Gamma}^\nu_{[\beta\omega]}] = 0 \tag{X.23}$$

Writing $\mathsf{g}_{\nu\mu} = \mathsf{g}^0_{\nu\mu}\tau_0 + \mathsf{g}^1_{\nu\mu}\tau_1 + \mathsf{g}^2_{\nu\mu}\tau_2 + \mathsf{g}^3_{\nu\mu}\tau_3$ and using (X.22,X.23) and the fact that $\tau_i\tau_j + \tau_j\tau_i = 0$ we see that

$$\mathsf{g}^0_{\lambda\rho}\tilde{\Gamma}^\rho_{[\beta\omega]} = 0. \tag{X.24}$$

Assuming that $\mathsf{g}^0_{\lambda\rho}$ is invertible we get

$$\tilde{\Gamma}^\rho_{[\beta\omega]} = 0. \tag{X.25}$$

So $\tilde{\Gamma}^\tau_{\nu\mu}$ must have a traceless part in $\tilde{\Gamma}^\tau_{(\nu\mu)}$ and not just in $\tilde{\Gamma}^\tau_{[\nu\mu]}$.

Now let us assume that $\tilde{\Gamma}^\alpha_{[\nu\mu]} = 0$. Setting $\delta\mathcal{L}/\delta\tilde{\Gamma}^\beta_{\tau\rho} = 0$ using (X.7) while using a Lagrange multiplier $tr[\Omega^{\nu\mu}_\alpha\tilde{\Gamma}^\alpha_{[\nu\mu]}]$ to enforce the symmetry and using $N^{\dashv[\nu\mu]} = 0$, $\mathcal{A}_\nu = 0$

and the metric definition (X.1) gives

$$
\begin{aligned}
0 &= -(\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau})_{,\,\beta} - \frac{1}{2}\tilde{\Gamma}^{\rho}_{\beta\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\tau} - \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\nu}\tilde{\Gamma}^{\tau}_{\nu\beta} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\beta\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\rho} - \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\beta} \\
&\quad + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau} + \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\beta\alpha} \\
&\quad + \frac{1}{2}\delta^{\rho}_{\beta}\left((\mathsf{g}^{1/2d}\mathsf{g}^{\omega\tau})_{,\,\omega} + \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} + \frac{1}{2}\tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\right) \\
&\quad + \frac{1}{2}\delta^{\tau}_{\beta}\left((\mathsf{g}^{1/2d}\mathsf{g}^{\omega\rho})_{,\,\omega} + \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\rho}_{\nu\mu} + \frac{1}{2}\tilde{\Gamma}^{\rho}_{\nu\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\right). \qquad \text{(X.26)}
\end{aligned}
$$

Contracting gives

$$
\begin{aligned}
0 &= -(\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau})_{,\,\rho} - \frac{1}{2}\tilde{\Gamma}^{\rho}_{\rho\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\tau} - \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\nu}\tilde{\Gamma}^{\tau}_{\nu\rho} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\rho\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\rho} - \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\rho} \\
&\quad + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\rho\alpha}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau} + \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\rho\alpha} \\
&\quad + \frac{(n+1)}{2}\left((\mathsf{g}^{1/2d}\mathsf{g}^{\omega\tau})_{,\,\omega} + \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} + \frac{1}{2}\tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\right) \qquad \text{(X.27)} \\
&= \frac{(n-1)}{2}\left((\mathsf{g}^{1/2d}\mathsf{g}^{\omega\tau})_{,\,\omega} + \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\tilde{\Gamma}^{\tau}_{\nu\mu} + \frac{1}{2}\tilde{\Gamma}^{\tau}_{\nu\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\nu}\right). \qquad \text{(X.28)}
\end{aligned}
$$

So we can rewrite the connection equations as

$$
\begin{aligned}
0 &= -(\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau})_{,\,\beta} - \frac{1}{2}\tilde{\Gamma}^{\rho}_{\beta\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\tau} - \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\nu}\tilde{\Gamma}^{\tau}_{\nu\beta} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\beta\mu}\mathsf{g}^{1/2d}\mathsf{g}^{\mu\rho} - \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\beta} \\
&\quad + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau} + \frac{1}{2}\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\beta\alpha}. \qquad \text{(X.29)}
\end{aligned}
$$

Contracting with $\mathsf{g}_{\rho\tau}$ and taking the trace gives

$$
\begin{aligned}
0 &= tr[-(\mathsf{g}^{1/2d}\mathsf{g}^{\rho\tau})_{,\beta}\mathsf{g}_{\tau\rho}] + (n-2)tr[\tilde{\Gamma}^{\rho}_{\beta\rho}]\mathsf{g}^{1/2d} \qquad \text{(X.30)} \\
&= -dn(\mathsf{g}^{1/2d})_{,\beta} - \mathsf{g}^{1/2d}tr[\mathsf{g}^{\rho\tau}{}_{,\beta}\mathsf{g}_{\tau\rho}] + (n-2)tr[\tilde{\Gamma}^{\rho}_{\beta\rho}]\mathsf{g}^{1/2d} \qquad \text{(X.31)} \\
&= (n-2)(-d(\mathsf{g}^{1/2d})_{,\beta} + \mathsf{g}^{1/2d}tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]). \qquad \text{(X.32)}
\end{aligned}
$$

Using this result the connection equations become

$$
\begin{aligned}
0 &= -\mathsf{g}^{\rho\tau}{}_{,\beta} - \frac{1}{2}\tilde{\Gamma}^{\rho}_{\beta\mu}\mathsf{g}^{\mu\tau} - \frac{1}{2}\mathsf{g}^{\rho\nu}\tilde{\Gamma}^{\tau}_{\nu\beta} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\beta\mu}\mathsf{g}^{\mu\rho} - \frac{1}{2}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\beta} \\
&\quad + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\mathsf{g}^{\rho\tau} + \frac{1}{2}\mathsf{g}^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\beta\alpha} - \mathsf{g}^{\rho\tau}tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d. \qquad \text{(X.33)}
\end{aligned}
$$

Multiplying on the left by $\mathsf{g}_{\omega\rho}$ and on the right by $\mathsf{g}_{\tau\lambda}$ gives

$$\begin{aligned} 0 &= \mathsf{g}_{\omega\lambda,\beta} - \frac{1}{2}\mathsf{g}_{\omega\rho}\tilde{\Gamma}^{\rho}_{\beta\lambda} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\omega\beta}\mathsf{g}_{\tau\lambda} - \frac{1}{2}\mathsf{g}_{\omega\rho}\tilde{\Gamma}^{\tau}_{\beta\mu}\mathsf{g}^{\mu\rho}\mathsf{g}_{\tau\lambda} - \frac{1}{2}\mathsf{g}_{\omega\rho}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\beta}\mathsf{g}_{\tau\lambda} \\ &\quad + \frac{1}{2}\mathsf{g}_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\mathsf{g}_{\omega\lambda} - \mathsf{g}_{\omega\lambda}tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d. \end{aligned} \tag{X.34}$$

Multiplying on the left by $\mathsf{g}_{\tau\lambda}$ and on the right by $\mathsf{g}_{\omega\rho}$ gives the same result. Combining the connection equations with their permutations gives a rather useless result.

$$\begin{aligned} 0 &= \Big(\mathsf{g}_{\omega\lambda,\beta} - \frac{1}{2}\mathsf{g}_{\omega\rho}\tilde{\Gamma}^{\rho}_{\beta\lambda} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\omega\beta}\mathsf{g}_{\tau\lambda} - \frac{1}{2}\mathsf{g}_{\omega\rho}\tilde{\Gamma}^{\tau}_{\beta\mu}\mathsf{g}^{\mu\rho}\mathsf{g}_{\tau\lambda} - \frac{1}{2}\mathsf{g}_{\omega\rho}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\beta}\mathsf{g}_{\tau\lambda} \\ &\quad + \frac{1}{2}\mathsf{g}_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\mathsf{g}_{\omega\lambda} - \mathsf{g}_{\omega\lambda}tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d\Big) \\ &- \Big(\mathsf{g}_{\beta\omega,\lambda} - \frac{1}{2}\mathsf{g}_{\beta\rho}\tilde{\Gamma}^{\rho}_{\lambda\omega} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\beta\lambda}\mathsf{g}_{\tau\omega} - \frac{1}{2}\mathsf{g}_{\beta\rho}\tilde{\Gamma}^{\tau}_{\lambda\mu}\mathsf{g}^{\mu\rho}\mathsf{g}_{\tau\omega} - \frac{1}{2}\mathsf{g}_{\beta\rho}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\lambda}\mathsf{g}_{\tau\omega} \\ &\quad + \frac{1}{2}\mathsf{g}_{\beta\omega}\tilde{\Gamma}^{\alpha}_{\lambda\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\lambda\alpha}\mathsf{g}_{\beta\omega} - \mathsf{g}_{\beta\omega}tr[\tilde{\Gamma}^{\alpha}_{\lambda\alpha}]/d\Big) \\ &- \Big(\mathsf{g}_{\lambda\beta,\omega} - \frac{1}{2}\mathsf{g}_{\lambda\rho}\tilde{\Gamma}^{\rho}_{\omega\beta} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\lambda\omega}\mathsf{g}_{\tau\beta} - \frac{1}{2}\mathsf{g}_{\lambda\rho}\tilde{\Gamma}^{\tau}_{\omega\mu}\mathsf{g}^{\mu\rho}\mathsf{g}_{\tau\beta} - \frac{1}{2}\mathsf{g}_{\lambda\rho}\mathsf{g}^{\tau\nu}\tilde{\Gamma}^{\rho}_{\nu\omega}\mathsf{g}_{\tau\beta} \\ &\quad + \frac{1}{2}\mathsf{g}_{\omega\beta}\tilde{\Gamma}^{\alpha}_{\omega\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\omega\alpha}\mathsf{g}_{\lambda\beta} - \mathsf{g}_{\lambda\beta}tr[\tilde{\Gamma}^{\alpha}_{\omega\alpha}]/d\Big). \end{aligned} \tag{X.35}$$

Now let us assume that $\mathsf{g}_{\nu\mu} = \gamma g_{\nu\mu}$ where $g_{\nu\mu}$ has no traceless components and γ is a matrix. Then (X.34) becomes

$$\begin{aligned} 0 &= \gamma g_{\omega\lambda,\beta} + \gamma_{,\beta}g_{\omega\lambda} - \frac{1}{2}\gamma g_{\omega\rho}\tilde{\Gamma}^{\rho}_{\beta\lambda} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\omega\beta}\gamma g_{\tau\lambda} - \frac{1}{2}\gamma\tilde{\Gamma}^{\tau}_{\beta\omega}g_{\tau\lambda} - \frac{1}{2}g_{\omega\rho}\tilde{\Gamma}^{\rho}_{\lambda\beta}\gamma \\ &\quad + \frac{1}{2}\gamma g_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\gamma g_{\omega\lambda} - \gamma g_{\omega\lambda}tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d. \end{aligned} \tag{X.36}$$

Combining this with its permutations gives

$$
\begin{aligned}
0 &= \Big(\gamma g_{\omega\lambda,\beta} + \gamma_{,\beta} g_{\omega\lambda} - \frac{1}{2}\gamma g_{\omega\rho}\tilde{\Gamma}^{\rho}_{\beta\lambda} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\omega\beta}\gamma g_{\tau\lambda} - \frac{1}{2}\gamma\tilde{\Gamma}^{\tau}_{\beta\omega} g_{\tau\lambda} - \frac{1}{2} g_{\omega\rho}\tilde{\Gamma}^{\rho}_{\lambda\beta}\gamma \\
&\quad + \frac{1}{2}\gamma g_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\gamma g_{\omega\lambda} - \gamma g_{\omega\lambda} tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d\Big) \\
&- \Big(\gamma g_{\beta\omega,\lambda} + \gamma_{,\lambda} g_{\beta\omega} - \frac{1}{2}\gamma g_{\beta\rho}\tilde{\Gamma}^{\rho}_{\lambda\omega} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\beta\lambda}\gamma g_{\tau\omega} - \frac{1}{2}\gamma\tilde{\Gamma}^{\tau}_{\lambda\beta} g_{\tau\omega} - \frac{1}{2} g_{\beta\rho}\tilde{\Gamma}^{\rho}_{\omega\lambda}\gamma \\
&\quad + \frac{1}{2}\gamma g_{\beta\omega}\tilde{\Gamma}^{\alpha}_{\lambda\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\lambda\alpha}\gamma g_{\beta\omega} - \gamma g_{\beta\omega} tr[\tilde{\Gamma}^{\alpha}_{\lambda\alpha}]/d\Big) \\
&- \Big(\gamma g_{\lambda\beta,\omega} + \gamma_{,\omega} g_{\lambda\beta} - \frac{1}{2}\gamma g_{\lambda\rho}\tilde{\Gamma}^{\rho}_{\omega\beta} - \frac{1}{2}\tilde{\Gamma}^{\tau}_{\lambda\omega}\gamma g_{\tau\beta} - \frac{1}{2}\gamma\tilde{\Gamma}^{\tau}_{\omega\lambda} g_{\tau\beta} - \frac{1}{2} g_{\lambda\rho}\tilde{\Gamma}^{\rho}_{\beta\omega}\gamma \\
&\quad + \frac{1}{2}\gamma g_{\lambda\beta}\tilde{\Gamma}^{\alpha}_{\omega\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\omega\alpha}\gamma g_{\lambda\beta} - \gamma g_{\lambda\beta} tr[\tilde{\Gamma}^{\alpha}_{\omega\alpha}]/d\Big) \qquad (X.37) \\
&= \gamma g_{\beta\rho}\tilde{\Gamma}^{\rho}_{\lambda\omega} + g_{\beta\rho}\tilde{\Gamma}^{\rho}_{\omega\lambda}\gamma \\
&\quad + \Big(\gamma g_{\omega\lambda,\beta} + \gamma_{,\beta} g_{\omega\lambda} + \frac{1}{2}\gamma g_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\beta\alpha}\gamma g_{\omega\lambda} - \gamma g_{\omega\lambda} tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d\Big) \\
&\quad - \Big(\gamma g_{\beta\omega,\lambda} + \gamma_{,\lambda} g_{\beta\omega} + \frac{1}{2}\gamma g_{\beta\omega}\tilde{\Gamma}^{\alpha}_{\lambda\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\lambda\alpha}\gamma g_{\beta\omega} - \gamma g_{\beta\omega} tr[\tilde{\Gamma}^{\alpha}_{\lambda\alpha}]/d\Big) \\
&\quad - \Big(\gamma g_{\lambda\beta,\omega} + \gamma_{,\omega} g_{\lambda\beta} + \frac{1}{2}\gamma g_{\lambda\beta}\tilde{\Gamma}^{\alpha}_{\omega\alpha} + \frac{1}{2}\tilde{\Gamma}^{\alpha}_{\omega\alpha}\gamma g_{\lambda\beta} - \gamma g_{\lambda\beta} tr[\tilde{\Gamma}^{\alpha}_{\omega\alpha}]/d\Big). \qquad (X.38)
\end{aligned}
$$

Now let us consider the special case where $\mathsf{g}_{\nu\mu}$ and $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ are composed of the identity matrix and only one of the τ_i matrices, either τ_1, τ_2 or τ_3. In this case $\mathsf{g}_{\nu\mu}$ and $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ commute, and the connection equations (X.34) simplify to

$$
0 = \mathsf{g}_{\omega\lambda,\beta} - \mathsf{g}_{\omega\rho}\tilde{\Gamma}^{\rho}_{\beta\lambda} - \tilde{\Gamma}^{\tau}_{\omega\beta}\mathsf{g}_{\tau\lambda} + \mathsf{g}_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} - \mathsf{g}_{\omega\lambda} tr[\tilde{\Gamma}^{\alpha}_{\beta\alpha}]/d. \qquad (X.39)
$$

Combining this with its permutations and using (X.32) gives

$$
\begin{aligned}
0 &= \Big(\mathrm{g}_{\omega\lambda,\beta} - \mathrm{g}_{\omega\rho}\tilde{\Gamma}^{\rho}_{\beta\lambda} - \tilde{\Gamma}^{\tau}_{\omega\beta}\mathrm{g}_{\tau\lambda} + \mathrm{g}_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} - \mathrm{g}_{\omega\lambda}\mathrm{g}^{2d}(\mathrm{g}^{1/2d})_{,\beta}\Big) \\
&- \Big(\mathrm{g}_{\beta\omega,\lambda} - \mathrm{g}_{\beta\rho}\tilde{\Gamma}^{\rho}_{\lambda\omega} - \tilde{\Gamma}^{\tau}_{\beta\lambda}\mathrm{g}_{\tau\omega} + \mathrm{g}_{\beta\omega}\tilde{\Gamma}^{\alpha}_{\lambda\alpha} - \mathrm{g}_{\beta\omega}\mathrm{g}^{2d}(\mathrm{g}^{1/2d})_{,\lambda}\Big) \\
&- \Big(\mathrm{g}_{\lambda\beta,\omega} - \mathrm{g}_{\lambda\rho}\tilde{\Gamma}^{\rho}_{\omega\beta} - \tilde{\Gamma}^{\tau}_{\lambda\omega}\mathrm{g}_{\tau\beta} + \mathrm{g}_{\lambda\beta}\tilde{\Gamma}^{\alpha}_{\omega\alpha} - \mathrm{g}_{\lambda\beta}\mathrm{g}^{2d}(\mathrm{g}^{1/2d})_{,\omega}\Big) \qquad \text{(X.40)} \\
&= 2\mathrm{g}_{\beta\rho}\tilde{\Gamma}^{\rho}_{\lambda\omega} + \Big(\mathrm{g}_{\omega\lambda,\beta} + \mathrm{g}_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} - \mathrm{g}_{\omega\lambda}tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\beta}]/2d\Big) \\
&\qquad - \Big(\mathrm{g}_{\beta\omega,\lambda} + \mathrm{g}_{\beta\omega}\tilde{\Gamma}^{\alpha}_{\lambda\alpha} - \mathrm{g}_{\beta\omega}tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\lambda}]/2d\Big) \\
&\qquad - \Big(\mathrm{g}_{\lambda\beta,\omega} + \mathrm{g}_{\lambda\beta}\tilde{\Gamma}^{\alpha}_{\omega\alpha} - \mathrm{g}_{\lambda\beta}tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\omega}]/2d\Big), \qquad \text{(X.41)} \\
\tilde{\Gamma}^{\rho}_{\lambda\omega} &= \frac{1}{2}\mathrm{g}^{\rho\beta}\Big(\mathrm{g}_{\beta\omega,\lambda} + \mathrm{g}_{\beta\omega}\tilde{\Gamma}^{\alpha}_{\lambda\alpha} - \mathrm{g}_{\beta\omega}tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\lambda}]/2d \\
&\qquad + \mathrm{g}_{\lambda\beta,\omega} + \mathrm{g}_{\lambda\beta}\tilde{\Gamma}^{\alpha}_{\omega\alpha} - \mathrm{g}_{\lambda\beta}tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\omega}]/2d \\
&\qquad - \mathrm{g}_{\omega\lambda,\beta} - \mathrm{g}_{\omega\lambda}\tilde{\Gamma}^{\alpha}_{\beta\alpha} + \mathrm{g}_{\omega\lambda}tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\beta}]/2d\Big). \qquad \text{(X.42)}
\end{aligned}
$$

Contracting this over ${}^{\rho}_{\omega}$ gives a generalization of the Christoffel connection gives

$$
\begin{aligned}
\tilde{\Gamma}^{\alpha}_{\lambda\alpha} &= \frac{1}{(2-n)}\big(\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\lambda} - n\,tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\lambda}]/2d\big), \qquad \text{(X.43)} \\
\tilde{\Gamma}^{\rho}_{\lambda\omega} &= \frac{1}{2}\mathrm{g}^{\rho\beta}\Big(\mathrm{g}_{\beta\omega,\lambda} + \frac{1}{(2-n)}\,\mathrm{g}_{\beta\omega}(\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\lambda} - tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\lambda}]/d) \\
&\qquad + \mathrm{g}_{\lambda\beta,\omega} + \frac{1}{(2-n)}\,\mathrm{g}_{\lambda\beta}(\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\omega} - tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\omega}]/d) \\
&\qquad - \mathrm{g}_{\omega\lambda,\beta} - \frac{1}{(2-n)}\,\mathrm{g}_{\omega\lambda}(\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\beta} - tr[\mathrm{g}^{\alpha\tau}\mathrm{g}_{\tau\alpha,\beta}]/d)\Big). \qquad \text{(X.44)}
\end{aligned}
$$

Now let us consider the case where the traceless components are small. We define

$$\mathrm{g}^{1/2d}\mathrm{g}^{\nu\mu} = g^{1/2d}(g^{\nu\mu} - \bar{h}^{\nu\mu}), \qquad \tilde{\Gamma}^{\alpha}_{\nu\mu} = \Gamma^{\alpha}_{\nu\mu} + H^{\alpha}_{\nu\mu}, \qquad \text{(X.45)}$$

where $\bar{h}^{\nu}_{\mu} \ll 1$, $g^{\nu\mu} = tr[g^{\nu\mu}]/d$, $\Gamma^{\alpha}_{\nu\mu}$ is the Christoffel connection formed from $g_{\nu\mu}$ and

$$tr[\bar{h}^{\nu\mu}] = 0. \qquad \text{(X.46)}$$

Lowering an index on the right side of (X.45) we get

$$(\mathsf{g}/g)^{1/2d}\mathsf{g}^{\mu}_{\alpha} = \delta^{\mu}_{\alpha}I - \bar{h}^{\mu}_{\alpha}. \tag{X.47}$$

Using the well known formula $det(e^M) = exp\,(tr(M))$, and the power series $ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4\ldots$ we get[85],

$$ln(det(I-\bar{h})) \;=\; tr[ln(I-\bar{h})] = -tr[\bar{h}^{\alpha}_{\alpha}] - \frac{1}{2}\,tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}] + \mathcal{O}(\bar{h}^3). \tag{X.48}$$

Taking $ln(det())$ on both sides of (X.47) using (X.48,X.46) and the identities $det(sM) = s^{nd}det(M)$ and $det(M^{-1}) = 1/det(M)$ gives

$$ln(det[(\mathsf{g}/g)^{1/2d}\mathsf{g}^{\mu}_{\alpha}]) = ln((\mathsf{g}/g)^{n/2-1}) = -\frac{1}{2}\,tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}] + \mathcal{O}(\bar{h}^3), \tag{X.49}$$

$$ln[(\mathsf{g}/g)^{1/2d}] = -\frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}]}{2d(n-2)} - \frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\tau}\bar{h}^{\tau}_{\rho}]}{3d(n-2)} - \frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\tau}\bar{h}^{\tau}_{\lambda}\bar{h}^{\lambda}_{\rho}]}{4d(n-2)} + \mathcal{O}(\bar{h}^5). \tag{X.50}$$

Taking e^x on both sides of this and using $e^x = 1 + x + x^2/2\ldots$ gives

$$(\mathsf{g}/g)^{1/2d} = 1 - \frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}]}{2d(n-2)} - \frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\tau}\bar{h}^{\tau}_{\rho}]}{3d(n-2)} - \frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\tau}\bar{h}^{\tau}_{\alpha}\bar{h}^{\alpha}_{\rho}]}{4d(n-2)} + \frac{(tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}])^2}{8d^2(n-2)^2} + \mathcal{O}(\bar{h}^5). \tag{X.51}$$

From (X.45) we see that the inverse field $\mathsf{g}_{\nu\mu}$ satisfies

$$\mathsf{g}_{\nu\mu} = (\mathsf{g}/g)^{1/2d}(g_{\nu\mu} + \bar{h}_{\nu\mu} + \bar{h}^{\alpha}_{\nu}\bar{h}_{\alpha\mu} + \bar{h}^{\alpha}_{\nu}\bar{h}^{\sigma}_{\alpha}\bar{h}_{\sigma\mu}) + \mathcal{O}(\bar{h}^4). \tag{X.52}$$

Let us also define the field $h_{\nu\mu}$ by

$$\mathsf{g}_{\nu\mu} = g_{\nu\mu} + h_{\nu\mu}. \tag{X.53}$$

Using (X.51,X.52) we can relate $h_{\nu\mu}$ and $\bar{h}_{\nu\mu}$,

$$\begin{aligned} h_{\nu\mu} \;=\; & \bar{h}_{\nu\mu} + \bar{h}^{\alpha}_{\nu}\bar{h}_{\alpha\mu} - g_{\nu\mu}\frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}]}{2d(n-2)} \\ & + \bar{h}^{\alpha}_{\nu}\bar{h}^{\sigma}_{\alpha}\bar{h}_{\sigma\mu} - \bar{h}_{\nu\mu}\frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}]}{2d(n-2)} - g_{\nu\mu}\frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\tau}\bar{h}^{\tau}_{\rho}]}{3d(n-2)} + \mathcal{O}(\bar{h}^4). \end{aligned} \tag{X.54}$$

Note that the definition of $\bar{h}_{\nu\mu}$ in (X.45) is the same as in linearized gravity with the substitution $\mathsf{g}_{\nu\mu} \to g_{\nu\mu}$, $g_{\nu\mu} \to \eta_{\nu\mu}$. However we do not get $h_{\nu\mu} \approx \bar{h}_{\nu\mu} - g_{\nu\mu}\bar{h}^{\tau}_{\tau}/(n-2)$ analogous to linearized gravity because the equivalent of $-g_{\nu\mu}\bar{h}^{\tau}_{\tau}/(n-2)$ would arise instead as $-g_{\nu\mu}tr[\bar{h}^{\tau}_{\tau}]/d(n-2)$ in (X.50), and this term vanishes because of (X.46). Nevertheless we will do all of our calculations using $\bar{h}_{\nu\mu}$ rather than $h_{\nu\mu}$, in which case with the substitution $\mathsf{g}_{\nu\mu} \to g_{\nu\mu}$, $g_{\nu\mu} \to \eta_{\nu\mu}$, all of our calculations will also apply for linearized gravity except for a few final results which use (X.51,X.54).

The connection equations (X.29) to $\mathcal{O}(\bar{h})$ are

$$\begin{aligned} 0 \;=\;& (g^{1/2d}\bar{h}^{\rho\tau})_{,\beta} + \frac{1}{2}\Gamma^{\rho}_{\beta\mu}g^{1/2d}\bar{h}^{\mu\tau} - \frac{1}{2}H^{\rho}_{\beta\mu}g^{1/2d}g^{\mu\tau} + \frac{1}{2}g^{1/2d}\bar{h}^{\rho\nu}\Gamma^{\tau}_{\nu\beta} - \frac{1}{2}g^{1/2d}g^{\rho\nu}H^{\tau}_{\nu\beta} \\ &+\frac{1}{2}\Gamma^{\tau}_{\beta\mu}g^{1/2d}\bar{h}^{\mu\rho} - \frac{1}{2}H^{\tau}_{\beta\mu}g^{1/2d}g^{\mu\rho} + \frac{1}{2}g^{1/2d}\bar{h}^{\tau\nu}\Gamma^{\rho}_{\nu\beta} - \frac{1}{2}g^{1/2d}g^{\tau\nu}H^{\rho}_{\nu\beta} \\ &-\frac{1}{2}\Gamma^{\alpha}_{\beta\alpha}g^{1/2d}\bar{h}^{\rho\tau} + \frac{1}{2}H^{\alpha}_{\beta\alpha}g^{1/2d}g^{\rho\tau} - \frac{1}{2}g^{1/2d}\bar{h}^{\rho\tau}\Gamma^{\alpha}_{\beta\alpha} + \frac{1}{2}g^{1/2d}g^{\rho\tau}H^{\alpha}_{\beta\alpha}. \end{aligned} \tag{X.55}$$

Using $I(g^{1/2d})_{,\beta} = g^{1/2d}\Gamma^{\alpha}_{\beta\alpha}$ and dividing by $g^{1/2d}$ gives

$$0 \;=\; \bar{h}^{\rho\tau}{}_{;\beta} - H^{\rho}_{\beta\mu}g^{\mu\tau} - g^{\rho\nu}H^{\tau}_{\nu\beta} + g^{\rho\tau}H^{\alpha}_{\beta\alpha}, \tag{X.56}$$

$$0 \;=\; \bar{h}_{\omega\lambda;\beta} - H_{\omega\beta\lambda} - H_{\lambda\omega\beta} + g_{\omega\lambda}H^{\alpha}_{\beta\alpha}. \tag{X.57}$$

Combining the permutations of this gives

$$\begin{aligned} 0 \;=\;& (\bar{h}_{\omega\lambda;\beta} - H_{\omega\beta\lambda} - H_{\lambda\omega\beta} + g_{\omega\lambda}H^{\alpha}_{\beta\alpha}) \\ &- (\bar{h}_{\beta\omega;\lambda} - H_{\beta\lambda\omega} - H_{\omega\beta\lambda} + g_{\beta\omega}H^{\alpha}_{\lambda\alpha}) \\ &- (\bar{h}_{\lambda\beta;\omega} - H_{\lambda\omega\beta} - H_{\beta\lambda\omega} + g_{\lambda\beta}H^{\alpha}_{\omega\alpha}) \end{aligned} \tag{X.58}$$

$$= 2H_{\beta\lambda\omega} + \bar{h}_{\omega\lambda;\beta} - \bar{h}_{\beta\omega;\lambda} - \bar{h}_{\lambda\beta;\omega} + g_{\omega\lambda}H^{\alpha}_{\beta\alpha} - g_{\beta\omega}H^{\alpha}_{\lambda\alpha} - g_{\lambda\beta}H^{\alpha}_{\omega\alpha}. \tag{X.59}$$

Contracting this with $g^{\beta\omega}$ and $g^{\lambda\omega}$ gives

$$
\begin{aligned}
0 &= 2H^{\omega}_{\lambda\omega} - \bar{h}^{\omega}_{\omega;\lambda} - nH^{\alpha}_{\lambda\alpha} \qquad \Rightarrow \qquad H^{\omega}_{\lambda\omega} = \frac{1}{(2-n)}\bar{h}^{\omega}_{\omega;\lambda}, &\text{(X.60)}\\
0 &= 2H_{\beta}{}^{\omega}{}_{\omega} + \bar{h}^{\omega}_{\omega;\beta} - 2\bar{h}^{\omega}_{\beta;\omega} + nH^{\alpha}_{\beta\alpha} - 2H^{\alpha}_{\beta\alpha} \qquad \Rightarrow \qquad H_{\beta}{}^{\omega}{}_{\omega} = \bar{h}^{\omega}_{\beta;\omega}. &\text{(X.61)}
\end{aligned}
$$

So the final result is

$$
H_{\alpha\nu\mu} = \frac{1}{2}(\bar{h}_{\alpha\nu;\mu} + \bar{h}_{\mu\alpha;\nu} - \bar{h}_{\nu\mu;\alpha}) + \frac{1}{2(2-n)}(g_{\alpha\nu}\bar{h}^{\omega}_{\omega;\mu} + g_{\mu\alpha}\bar{h}^{\omega}_{\omega;\nu} - g_{\nu\mu}\bar{h}^{\omega}_{\omega;\alpha}). \qquad \text{(X.62)}
$$

To find the $\mathcal{O}(\bar{h}^2)$ solution we assume that $\tilde{\Gamma}^{\alpha}_{\nu\mu} = I\Gamma^{\alpha}_{\nu\mu} + H^{\alpha}_{\nu\mu} + K^{\alpha}_{\nu\mu}$ and solve the connection equations (X.29) to $\mathcal{O}(\bar{h}^2)$,

$$
\begin{aligned}
0 &= -\frac{1}{2}K^{\rho}_{\beta\mu}g^{\mu\tau} + \frac{1}{2}H^{\rho}_{\beta\mu}\bar{h}^{\mu\tau} - \frac{1}{2}g^{\rho\nu}K^{\tau}_{\nu\beta} + \frac{1}{2}\bar{h}^{\rho\nu}H^{\tau}_{\nu\beta}\\
&\quad -\frac{1}{2}K^{\tau}_{\beta\mu}g^{\mu\rho} + \frac{1}{2}H^{\tau}_{\beta\mu}\bar{h}^{\mu\rho} - \frac{1}{2}g^{\tau\nu}K^{\rho}_{\nu\beta} + \frac{1}{2}\bar{h}^{\tau\nu}H^{\rho}_{\nu\beta}\\
&\quad +\frac{1}{2}K^{\alpha}_{\beta\alpha}g^{\rho\tau} - \frac{1}{2}H^{\alpha}_{\beta\alpha}\bar{h}^{\rho\tau} + \frac{1}{2}g^{\rho\tau}K^{\alpha}_{\beta\alpha} - \frac{1}{2}\bar{h}^{\rho\tau}H^{\alpha}_{\beta\alpha} &\text{(X.63)}\\
&= -K^{\rho}_{\beta\mu}g^{\mu\tau} - g^{\rho\nu}K^{\tau}_{\nu\beta} + g^{\rho\tau}K^{\alpha}_{\beta\alpha}\\
&\quad +\frac{1}{2}(H^{\rho}_{\beta\mu}\bar{h}^{\mu\tau} + \bar{h}^{\rho\nu}H^{\tau}_{\nu\beta} + H^{\tau}_{\beta\mu}\bar{h}^{\mu\rho} + \bar{h}^{\tau\nu}H^{\rho}_{\nu\beta} - H^{\alpha}_{\beta\alpha}\bar{h}^{\rho\tau} - \bar{h}^{\rho\tau}H^{\alpha}_{\beta\alpha}), &\text{(X.64)}\\
0 &= -K_{\rho\beta\tau} - K_{\tau\rho\beta} + g_{\rho\tau}K^{\alpha}_{\beta\alpha}\\
&\quad +\frac{1}{2}(H_{\rho\beta\mu}\bar{h}^{\mu}_{\tau} + \bar{h}^{\nu}_{\rho}H_{\tau\nu\beta} + H_{\tau\beta\mu}\bar{h}^{\mu}_{\rho} + \bar{h}^{\nu}_{\tau}H_{\rho\nu\beta} - H^{\alpha}_{\beta\alpha}\bar{h}_{\rho\tau} - \bar{h}_{\rho\tau}H^{\alpha}_{\beta\alpha}). &\text{(X.65)}
\end{aligned}
$$

Combining the permutations of this gives

$$
\begin{aligned}
0 \;=\;& -K_{\rho\beta\tau} - K_{\tau\rho\beta} + g_{\rho\tau}K^{\alpha}_{\beta\alpha} \\
& +\frac{1}{2}(H_{\rho\beta\mu}\bar{h}^{\mu}_{\tau} + \bar{h}^{\nu}_{\rho}H_{\tau\nu\beta} + H_{\tau\beta\mu}\bar{h}^{\mu}_{\rho} + \bar{h}^{\nu}_{\tau}H_{\rho\nu\beta} - H^{\alpha}_{\beta\alpha}\bar{h}_{\rho\tau} - \bar{h}_{\rho\tau}H^{\alpha}_{\beta\alpha}) \\
& +K_{\beta\tau\rho} + K_{\rho\beta\tau} - g_{\beta\rho}K^{\alpha}_{\tau\alpha} \\
& -\frac{1}{2}(H_{\beta\tau\mu}\bar{h}^{\mu}_{\rho} + \bar{h}^{\nu}_{\beta}H_{\rho\nu\tau} + H_{\rho\tau\mu}\bar{h}^{\mu}_{\beta} + \bar{h}^{\nu}_{\rho}H_{\beta\nu\tau} - H^{\alpha}_{\tau\alpha}\bar{h}_{\beta\rho} - \bar{h}_{\beta\rho}H^{\alpha}_{\tau\alpha}) \\
& +K_{\tau\rho\beta} + K_{\beta\tau\rho} - g_{\tau\beta}K^{\alpha}_{\rho\alpha} \\
& -\frac{1}{2}(H_{\tau\rho\mu}\bar{h}^{\mu}_{\beta} + \bar{h}^{\nu}_{\tau}H_{\beta\nu\rho} + H_{\beta\rho\mu}\bar{h}^{\mu}_{\tau} + \bar{h}^{\nu}_{\beta}H_{\tau\nu\rho} - H^{\alpha}_{\rho\alpha}\bar{h}_{\tau\beta} - \bar{h}_{\tau\beta}H^{\alpha}_{\rho\alpha}) \qquad \text{(X.66)} \\
=\;& 2K_{\beta\tau\rho} + g_{\rho\tau}K^{\alpha}_{\beta\alpha} - g_{\beta\rho}K^{\alpha}_{\tau\alpha} - g_{\tau\beta}K^{\alpha}_{\rho\alpha} \\
& +\frac{1}{2}(\; H_{\rho\sigma\beta}\bar{h}^{\sigma}_{\tau} + \bar{h}^{\sigma}_{\tau}H_{\rho\sigma\beta} - H_{\beta\sigma\tau}\bar{h}^{\sigma}_{\rho} - \bar{h}^{\sigma}_{\rho}H_{\beta\sigma\tau} - H_{\tau\sigma\rho}\bar{h}^{\sigma}_{\beta} - \bar{h}^{\sigma}_{\beta}H_{\tau\sigma\rho} \\
& \quad + H_{\tau\sigma\beta}\bar{h}^{\sigma}_{\rho} + \bar{h}^{\sigma}_{\rho}H_{\tau\sigma\beta} - H_{\beta\sigma\rho}\bar{h}^{\sigma}_{\tau} - \bar{h}^{\sigma}_{\tau}H_{\beta\sigma\rho} - H_{\rho\sigma\tau}\bar{h}^{\sigma}_{\beta} - \bar{h}^{\sigma}_{\beta}H_{\rho\sigma\tau} \\
& \quad - H^{\alpha}_{\beta\alpha}\bar{h}_{\rho\tau} - \bar{h}_{\rho\tau}H^{\alpha}_{\beta\alpha} + H^{\alpha}_{\tau\alpha}\bar{h}_{\beta\rho} + \bar{h}_{\beta\rho}H^{\alpha}_{\tau\alpha} + H^{\alpha}_{\rho\alpha}\bar{h}_{\tau\beta} + \bar{h}_{\tau\beta}H^{\alpha}_{\rho\alpha}). \qquad \text{(X.67)}
\end{aligned}
$$

Contracting this with $g^{\rho\beta}$ gives

$$0 \;=\; (2-n)K^{\rho}_{\tau\rho} - \bar{h}^{\nu}_{\rho}H^{\rho}_{\nu\tau} + \frac{1}{2}\bar{h}^{\rho}_{\rho}H^{\alpha}_{\tau\alpha} - H^{\rho}_{\tau\mu}\bar{h}^{\mu}_{\rho} + \frac{1}{2}H^{\alpha}_{\tau\alpha}\bar{h}^{\rho}_{\rho}. \qquad \text{(X.68)}$$

This can be simplified by substituting (X.62),

$$\bar{h}^{\alpha\nu}H_{\alpha\nu\mu} - \frac{1}{2}\bar{h}^{\rho}_{\rho}H^{\alpha}_{\mu\alpha} = \frac{1}{2}\bar{h}^{\alpha\nu}\bar{h}_{\alpha\nu;\mu} + \frac{1}{2(2-n)}\bar{h}^{\alpha\nu}g_{\alpha\nu}\bar{h}^{\tau}_{\tau;\mu} - \frac{1}{2}\bar{h}^{\rho}_{\rho}\frac{1}{(2-n)}\bar{h}^{\alpha}_{\alpha;\mu} \qquad \text{(X.69)}$$

$$= \bar{h}^{\alpha\nu}\bar{h}_{\nu\alpha;\mu}/2, \qquad \text{(X.70)}$$

$$H_{\alpha\nu\mu}\bar{h}^{\alpha\nu} - \frac{1}{2}H^{\alpha}_{\mu\alpha}\bar{h}^{\rho}_{\rho} = \frac{1}{2}\bar{h}_{\alpha\nu;\mu}\bar{h}^{\alpha\nu} + \frac{1}{2(2-n)}g_{\alpha\nu}\bar{h}^{\tau}_{\tau;\mu}\bar{h}^{\alpha\nu} - \frac{1}{2}\frac{1}{(2-n)}\bar{h}^{\alpha}_{\alpha;\mu}\bar{h}^{\rho}_{\rho} \qquad \text{(X.71)}$$

$$= \bar{h}_{\nu\alpha;\mu}\bar{h}^{\alpha\nu}/2, \qquad \text{(X.72)}$$

$$\Rightarrow \quad 0 = (2-n)K^{\rho}_{\tau\rho} - \frac{1}{2}(\bar{h}^{\alpha}_{\nu}\bar{h}^{\nu}_{\alpha})_{,\tau} \quad \Rightarrow \quad K^{\rho}_{\tau\rho} = \frac{1}{2(2-n)}(\bar{h}^{\alpha}_{\nu}\bar{h}^{\nu}_{\alpha})_{,\tau}. \qquad \text{(X.73)}$$

Using this last result and (X.62,X.60) we can simplify (X.67),

$$
\begin{aligned}
K_{\beta\tau\rho} &= \frac{1}{2}(-g_{\rho\tau}K^{\alpha}_{\beta\alpha}+g_{\beta\rho}K^{\alpha}_{\tau\alpha}+g_{\tau\beta}K^{\alpha}_{\rho\alpha}) \\
&+ \frac{1}{4}(-H_{\rho\sigma\beta}\bar{h}^{\sigma}_{\tau}-\bar{h}^{\sigma}_{\tau}H_{\rho\sigma\beta}+H_{\beta\sigma\tau}\bar{h}^{\sigma}_{\rho}+\bar{h}^{\sigma}_{\rho}H_{\beta\sigma\tau}+H_{\tau\sigma\rho}\bar{h}^{\sigma}_{\beta}+\bar{h}^{\sigma}_{\beta}H_{\tau\sigma\rho} \\
&\quad -H_{\tau\sigma\beta}\bar{h}^{\sigma}_{\rho}-\bar{h}^{\sigma}_{\rho}H_{\tau\sigma\beta}+H_{\beta\sigma\rho}\bar{h}^{\sigma}_{\tau}+\bar{h}^{\sigma}_{\tau}H_{\beta\sigma\rho}+H_{\rho\sigma\tau}\bar{h}^{\sigma}_{\beta}+\bar{h}^{\sigma}_{\beta}H_{\rho\sigma\tau} \\
&\quad +H^{\alpha}_{\beta\alpha}\bar{h}_{\rho\tau}+\bar{h}_{\rho\tau}H^{\alpha}_{\beta\alpha}-H^{\alpha}_{\tau\alpha}\bar{h}_{\beta\rho}-\bar{h}_{\beta\rho}H^{\alpha}_{\tau\alpha}-H^{\alpha}_{\rho\alpha}\bar{h}_{\tau\beta}-\bar{h}_{\tau\beta}H^{\alpha}_{\rho\alpha}) \qquad \text{(X.74)} \\
&= \frac{1}{4(2-n)}(-g_{\rho\tau}(\bar{h}^{\alpha}_{\nu}\bar{h}^{\nu}_{\alpha})_{,\beta}+g_{\beta\rho}(\bar{h}^{\alpha}_{\nu}\bar{h}^{\nu}_{\alpha})_{,\tau}+g_{\tau\beta}(\bar{h}^{\alpha}_{\nu}\bar{h}^{\nu}_{\alpha})_{,\rho}) \\
&- \frac{1}{8}(\bar{h}_{\rho\sigma;\beta}+\bar{h}_{\beta\rho;\sigma}-\bar{h}_{\sigma\beta;\rho})\bar{h}^{\sigma}_{\tau}-\frac{1}{8(2-n)}(g_{\rho\sigma}\bar{h}^{\omega}_{\omega;\beta}+g_{\beta\rho}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\beta}\bar{h}^{\omega}_{\omega;\rho})\bar{h}^{\sigma}_{\tau} \\
&- \frac{1}{8}\bar{h}^{\sigma}_{\tau}(\bar{h}_{\rho\sigma;\beta}+\bar{h}_{\beta\rho;\sigma}-\bar{h}_{\sigma\beta;\rho})-\frac{1}{8(2-n)}\bar{h}^{\sigma}_{\tau}(g_{\rho\sigma}\bar{h}^{\omega}_{\omega;\beta}+g_{\beta\rho}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\beta}\bar{h}^{\omega}_{\omega;\rho}) \\
&+ \frac{1}{8}(\bar{h}_{\beta\sigma;\tau}+\bar{h}_{\tau\beta;\sigma}-\bar{h}_{\sigma\tau;\beta})\bar{h}^{\sigma}_{\rho}+\frac{1}{8(2-n)}(g_{\beta\sigma}\bar{h}^{\omega}_{\omega;\tau}+g_{\tau\beta}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\tau}\bar{h}^{\omega}_{\omega;\beta})\bar{h}^{\sigma}_{\rho} \\
&+ \frac{1}{8}\bar{h}^{\sigma}_{\rho}(\bar{h}_{\beta\sigma;\tau}+\bar{h}_{\tau\beta;\sigma}-\bar{h}_{\sigma\tau;\beta})+\frac{1}{8(2-n)}\bar{h}^{\sigma}_{\rho}(g_{\beta\sigma}\bar{h}^{\omega}_{\omega;\tau}+g_{\tau\beta}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\tau}\bar{h}^{\omega}_{\omega;\beta}) \\
&+ \frac{1}{8}(\bar{h}_{\tau\sigma;\rho}+\bar{h}_{\rho\tau;\sigma}-\bar{h}_{\sigma\rho;\tau})\bar{h}^{\sigma}_{\beta}+\frac{1}{8(2-n)}(g_{\tau\sigma}\bar{h}^{\omega}_{\omega;\rho}+g_{\rho\tau}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\rho}\bar{h}^{\omega}_{\omega;\tau})\bar{h}^{\sigma}_{\beta} \\
&+ \frac{1}{8}\bar{h}^{\sigma}_{\beta}(\bar{h}_{\tau\sigma;\rho}+\bar{h}_{\rho\tau;\sigma}-\bar{h}_{\sigma\rho;\tau})+\frac{1}{8(2-n)}\bar{h}^{\sigma}_{\beta}(g_{\tau\sigma}\bar{h}^{\omega}_{\omega;\rho}+g_{\rho\tau}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\rho}\bar{h}^{\omega}_{\omega;\tau}) \\
&- \frac{1}{8}(\bar{h}_{\tau\sigma;\beta}+\bar{h}_{\beta\tau;\sigma}-\bar{h}_{\sigma\beta;\tau})\bar{h}^{\sigma}_{\rho}-\frac{1}{8(2-n)}(g_{\tau\sigma}\bar{h}^{\omega}_{\omega;\beta}+g_{\beta\tau}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\beta}\bar{h}^{\omega}_{\omega;\tau})\bar{h}^{\sigma}_{\rho} \\
&- \frac{1}{8}\bar{h}^{\sigma}_{\rho}(\bar{h}_{\tau\sigma;\beta}+\bar{h}_{\beta\tau;\sigma}-\bar{h}_{\sigma\beta;\tau})-\frac{1}{8(2-n)}\bar{h}^{\sigma}_{\rho}(g_{\tau\sigma}\bar{h}^{\omega}_{\omega;\beta}+g_{\beta\tau}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\beta}\bar{h}^{\omega}_{\omega;\tau}) \\
&+ \frac{1}{8}(\bar{h}_{\beta\sigma;\rho}+\bar{h}_{\rho\beta;\sigma}-\bar{h}_{\sigma\rho;\beta})\bar{h}^{\sigma}_{\tau}+\frac{1}{8(2-n)}(g_{\beta\sigma}\bar{h}^{\omega}_{\omega;\rho}+g_{\rho\beta}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\rho}\bar{h}^{\omega}_{\omega;\beta})\bar{h}^{\sigma}_{\tau} \\
&+ \frac{1}{8}\bar{h}^{\sigma}_{\tau}(\bar{h}_{\beta\sigma;\rho}+\bar{h}_{\rho\beta;\sigma}-\bar{h}_{\sigma\rho;\beta})+\frac{1}{8(2-n)}\bar{h}^{\sigma}_{\tau}(g_{\beta\sigma}\bar{h}^{\omega}_{\omega;\rho}+g_{\rho\beta}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\rho}\bar{h}^{\omega}_{\omega;\beta}) \\
&+ \frac{1}{8}(\bar{h}_{\rho\sigma;\tau}+\bar{h}_{\tau\rho;\sigma}-\bar{h}_{\sigma\tau;\rho})\bar{h}^{\sigma}_{\beta}+\frac{1}{8(2-n)}(g_{\rho\sigma}\bar{h}^{\omega}_{\omega;\tau}+g_{\tau\rho}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\tau}\bar{h}^{\omega}_{\omega;\rho})\bar{h}^{\sigma}_{\beta} \\
&+ \frac{1}{8}\bar{h}^{\sigma}_{\beta}(\bar{h}_{\rho\sigma;\tau}+\bar{h}_{\tau\rho;\sigma}-\bar{h}_{\sigma\tau;\rho})+\frac{1}{8(2-n)}\bar{h}^{\sigma}_{\beta}(g_{\rho\sigma}\bar{h}^{\omega}_{\omega;\tau}+g_{\tau\rho}\bar{h}^{\omega}_{\omega;\sigma}-g_{\sigma\tau}\bar{h}^{\omega}_{\omega;\rho}) \\
&+ \frac{1}{4(2-n)}(\bar{h}^{\alpha}_{\alpha;\beta}\bar{h}_{\rho\tau}+\bar{h}_{\rho\tau}\bar{h}^{\alpha}_{\alpha;\beta}-\bar{h}^{\alpha}_{\alpha;\tau}\bar{h}_{\beta\rho}-\bar{h}_{\beta\rho}\bar{h}^{\alpha}_{\alpha;\tau}-\bar{h}^{\alpha}_{\alpha;\rho}\bar{h}_{\tau\beta}-\bar{h}_{\tau\beta}\bar{h}^{\alpha}_{\alpha;\rho}), \qquad \text{(X.75)}
\end{aligned}
$$

$$
\begin{aligned}
K_{\beta\tau\rho} \;=\; & \frac{1}{4(2-n)}\,[-g_{\rho\tau}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{,\beta} + g_{\beta\rho}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{,\tau} + g_{\tau\beta}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{,\rho} \\
& \quad + g_{\rho\tau}(\bar{h}^{\omega}_{\omega;\sigma}\bar{h}^{\sigma}_{\beta} + \bar{h}^{\sigma}_{\beta}\bar{h}^{\omega}_{\omega;\sigma}) - \bar{h}^{\omega}_{\omega;\beta}\bar{h}_{\rho\tau} - \bar{h}_{\rho\tau}\bar{h}^{\omega}_{\omega;\beta}] \\
& + \frac{1}{4}[(\bar{h}_{\sigma\beta;\rho} - \bar{h}_{\rho\sigma;\beta})\bar{h}^{\sigma}_{\tau} + \bar{h}^{\sigma}_{\tau}(\bar{h}_{\sigma\beta;\rho} - \bar{h}_{\rho\sigma;\beta}) \\
& \quad + (\bar{h}_{\beta\sigma;\tau} - \bar{h}_{\sigma\tau;\beta})\bar{h}^{\sigma}_{\rho} + \bar{h}^{\sigma}_{\rho}(\bar{h}_{\beta\sigma;\tau} - \bar{h}_{\sigma\tau;\beta}) + \bar{h}_{\rho\tau;\sigma}\bar{h}^{\sigma}_{\beta} + \bar{h}^{\sigma}_{\beta}\bar{h}_{\rho\tau;\sigma}]. \quad \text{(X.76)}
\end{aligned}
$$

This result can be checked by comparing its contraction over $g^{\beta\rho}$ to (X.73). It can also be checked by comparing its contraction over $g^{\tau\rho}$ to the contraction of (X.65),

$$
K_{\alpha}{}^{\nu}{}_{\nu} = \frac{1}{2}(\bar{h}^{\mu\nu}H_{\alpha\nu\mu} + H_{\alpha\nu\mu}\bar{h}^{\mu\nu}) \quad \text{(X.77)}
$$

$$
\begin{aligned}
= \; & \frac{1}{2}\bar{h}^{\mu\nu}\left(\frac{1}{2}(2\bar{h}_{\alpha\nu;\mu} - \bar{h}_{\nu\mu;\alpha}) + \frac{1}{2(2-n)}(2g_{\alpha\nu}\bar{h}^{\omega}_{\omega;\mu} - g_{\nu\mu}\bar{h}^{\omega}_{\omega;\alpha})\right) \\
& + \frac{1}{2}\left(\frac{1}{2}(2\bar{h}_{\alpha\nu;\mu} - \bar{h}_{\nu\mu;\alpha}) + \frac{1}{2(2-n)}(2g_{\alpha\nu}\bar{h}^{\omega}_{\omega;\mu} - g_{\nu\mu}\bar{h}^{\omega}_{\omega;\alpha})\right)\bar{h}^{\mu\nu} \quad \text{(X.78)}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{1}{2}(\bar{h}^{\mu\nu}\bar{h}_{\alpha\nu;\mu} + \bar{h}_{\alpha\nu;\mu}\bar{h}^{\mu\nu}) - \frac{1}{4}(\bar{h}^{\mu\nu}\bar{h}_{\nu\mu})_{,\alpha} \\
& - \frac{1}{2(n-2)}(\bar{h}^{\mu}_{\alpha}\bar{h}^{\omega}_{\omega;\mu} + \bar{h}^{\omega}_{\omega;\mu}\bar{h}^{\mu}_{\alpha}) + \frac{1}{4(n-2)}(\bar{h}^{\nu}_{\nu}\bar{h}^{\omega}_{\omega})_{,\alpha}. \quad \text{(X.79)}
\end{aligned}
$$

Now let us calculate a weak field $\mathcal{L}_m$ term for $\bar{h}_{\nu\mu}$, assuming symmetric fields. Substituting our solution for $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ into (R.4), we find that because of the trace operation there are no $\mathcal{O}(\bar{h})$ terms of (7.20). Using (R.4,X.77,X.51,X.46), the $\mathcal{O}(\bar{h}^2)$ terms of

(7.20) are

$$\mathcal{L}_h = -\frac{1}{16\pi}\, g^{1/2d} tr[\quad g^{\nu\mu}(K^{\alpha}_{\nu\mu;\alpha} - K^{\alpha}_{\alpha(\nu;\mu)} + H^{\sigma}_{\nu\mu}H^{\alpha}_{\sigma\alpha} - H^{\sigma}_{\nu\alpha}H^{\alpha}_{\sigma\mu})$$

$$- \bar{h}^{\nu\mu}(H^{\alpha}_{\nu\mu;\alpha} - H^{\alpha}_{\alpha(\nu;\mu)})]$$

$$-\frac{1}{16\pi}\,[\,\mathsf{g}^{1/2d}d(n-2)\Lambda_b - g^{1/2d}d(n-2)\Lambda_b] \qquad \text{(X.80)}$$

$$= -\frac{1}{16\pi}\, g^{1/2d} tr[(\bar{h}^{\mu\nu}H^{\alpha}_{\nu\mu})_{;\alpha} - K^{\alpha}_{\alpha\nu;}{}^{\nu} + H^{\sigma\nu}{}_{\nu}H^{\alpha}_{\sigma\alpha} - H^{\sigma\nu}{}_{\alpha}H^{\alpha}_{\sigma\nu}$$

$$- \bar{h}^{\nu\mu}H^{\alpha}_{\nu\mu;\alpha} + \bar{h}^{\nu\mu}H^{\alpha}_{\alpha\nu;\mu} - \bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}\Lambda_b/2] \qquad \text{(X.81)}$$

$$= \frac{1}{16\pi}\, g^{1/2d} tr[\,\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}\Lambda_b/2 - \bar{h}^{\mu\nu}{}_{;\alpha}H^{\alpha}_{\nu\mu} + H^{\sigma\nu}{}_{\alpha}H^{\alpha}_{\sigma\nu}$$

$$+ K^{\alpha}_{\alpha\nu;}{}^{\nu} - H^{\sigma\nu}{}_{\nu}H^{\alpha}_{\sigma\alpha} - \bar{h}^{\nu\mu}H^{\alpha}_{\alpha\nu;\mu}]. \qquad \text{(X.82)}$$

Using (X.62,X.60,X.61,X.73) gives

$$\mathcal{L}_h = \frac{g^{1/2d}}{16\pi}\, tr\Bigg[\frac{\Lambda_b}{2}\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho} - \bar{h}^{\mu\nu}{}_{;\alpha}\left(\frac{1}{2}(2\bar{h}^{\alpha}_{\nu;\mu} - \bar{h}_{\nu\mu;}{}^{\alpha}) + \frac{1}{2(2-n)}(2\delta^{\alpha}_{\nu}\bar{h}^{\omega}_{\omega;\mu} - g_{\nu\mu}\bar{h}^{\omega}_{\omega;}{}^{\alpha})\right)$$

$$+ \left(\frac{1}{2}\bar{h}^{\sigma\nu}{}_{;\alpha} + \frac{1}{2(2-n)}g^{\sigma\nu}\bar{h}^{\omega}_{\omega;\alpha}\right)$$

$$\times\left(\frac{1}{2}(\bar{h}^{\alpha}_{\sigma;\nu} + \bar{h}^{\alpha}_{\nu;\sigma} - \bar{h}_{\sigma\nu;}{}^{\alpha}) + \frac{1}{2(2-n)}(\delta^{\alpha}_{\sigma}\bar{h}^{\omega}_{\omega;\nu} + \delta^{\alpha}_{\nu}\bar{h}^{\omega}_{\omega;\sigma} - g_{\sigma\nu}\bar{h}^{\omega}_{\omega;}{}^{\alpha})\right)$$

$$+\frac{1}{2(2-n)}(\bar{h}^{\alpha}_{\tau}\bar{h}^{\tau}_{\alpha})_{,\nu;}{}^{\nu} - \bar{h}^{\sigma\nu}{}_{;\nu}\frac{1}{(2-n)}\bar{h}^{\alpha}_{\alpha;\sigma} - \bar{h}^{\nu\mu}\frac{1}{(2-n)}\bar{h}^{\alpha}_{\alpha;\nu;\mu}\Bigg] \qquad \text{(X.83)}$$

$$= \frac{g^{1/2d}}{16\pi}\, tr[\,\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}\Lambda_b/2$$

$$+ \bar{h}^{\mu\nu}{}_{;\alpha}\bar{h}^{\alpha}_{\nu;\mu}(-1 + 1/4 + 1/4)$$

$$+ \bar{h}^{\mu\nu}{}_{;\alpha}\bar{h}_{\nu\mu;}{}^{\alpha}(1/2 - 1/4 + 1/2(2-n) + 1/2(2-n))$$

$$+ \bar{h}^{\mu\nu}{}_{;\nu}\bar{h}^{\omega}_{\omega;\mu}(-1 + 1/4 + 1/4 + 1/4 + 1/4 - 1)/(2-n)$$

$$+ \bar{h}^{\nu}_{\nu;}{}^{\mu}\bar{h}^{\omega}_{\omega;\mu}((1/2 - 1/4 - 1/4)/(2-n) + (1/4 + 1/4 - n/4)/(2-n)^2)$$

$$+ \bar{h}^{\nu\mu}\bar{h}_{\nu\mu;}{}^{\alpha}{}_{;\alpha}(1/2 + 1/2)/(2-n)$$

$$+ \bar{h}^{\nu\mu}\bar{h}^{\alpha}_{\alpha;\nu;\mu}(-1)/(2-n)\,] \qquad \text{(X.84)}$$

$$
\begin{aligned}
= \; & \frac{g^{1/2d}}{16\pi}\, tr[\, \bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}\Lambda_b/2 \\
& \qquad - \bar{h}^{\mu\nu}{}_{;\alpha}\bar{h}^{\alpha}_{\nu;\mu}/2 \\
& \qquad + \bar{h}^{\mu\nu}{}_{;\alpha}\bar{h}_{\nu\mu;}{}^{\alpha}(1/4 - 1/(n-2)) \\
& \qquad + \bar{h}^{\mu\nu}{}_{;\nu}\bar{h}^{\omega}_{\omega;\mu}/(n-2) \\
& \qquad - \bar{h}^{\nu}_{\nu;}{}^{\mu}\bar{h}^{\omega}_{\omega;\mu}/4(n-2) \\
& \qquad - \bar{h}^{\nu\mu}\bar{h}_{\nu\mu;}{}^{\alpha}{}_{;\alpha}/(n-2) \\
& \qquad + \bar{h}^{\nu\mu}\bar{h}^{\alpha}_{\alpha;\nu;\mu}/(n-2)\,] \qquad \text{(X.85)}\\
= \; & \frac{g^{1/2d}}{16\pi}\, tr[-\bar{h}^{\mu\nu}{}_{;\alpha}\bar{h}^{\alpha}_{\nu;\mu}/2 + \bar{h}^{\mu\nu}{}_{;\alpha}\bar{h}_{\nu\mu;}{}^{\alpha}/4 - \bar{h}^{\nu}_{\nu;}{}^{\mu}\bar{h}^{\omega}_{\omega;\mu}/4(n-2) + \bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}\Lambda_b/2 \\
& \qquad - (\bar{h}^{\nu}_{\mu}\bar{h}^{\mu}_{\nu})_{;}{}^{\alpha}{}_{;\alpha}/2(n-2) + (\bar{h}^{\nu\mu}\bar{h}^{\alpha}_{\alpha;\nu})_{;\mu}/(n-2)\,]. \qquad \text{(X.86)}
\end{aligned}
$$

The total divergence terms on the second line have no effect on the field equations and can be removed. So the final weak field Lagrangian density is

$$
\mathcal{L}_h \;=\; \frac{g^{1/2d}}{32\pi}\, tr\left[\frac{1}{2}\bar{h}^{\nu}_{\mu;\alpha}\bar{h}^{\mu}_{\nu;}{}^{\alpha} - \bar{h}^{\mu\nu}{}_{;\alpha}\bar{h}^{\alpha}_{\nu;\mu} - \frac{1}{2(n-2)}\bar{h}^{\nu}_{\nu;}{}^{\mu}\bar{h}^{\omega}_{\omega;\mu} + \bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}\Lambda_b\right]. \qquad \text{(X.87)}
$$

The field equations for $\bar{h}_{\nu\mu}$ can be found from the Lagrangian density (X.87),

$$
\begin{aligned}
0 \;=\; & 32\pi\frac{\delta(\mathcal{L}_h/g^{1/2d})}{\delta\bar{h}^{\rho\tau}} = 32\pi\frac{\partial(\mathcal{L}_h/g^{1/2d})}{\partial\bar{h}^{\rho\tau}} - 32\pi\left(\frac{\partial(\mathcal{L}_h/g^{1/2d})}{\partial(\bar{h}^{\rho\tau}{}_{;\sigma})}\right)_{;\sigma} \qquad \text{(X.88)}\\
=\; & \left(-\frac{2}{2}\delta^{\mu}_{\rho}\delta^{\nu}_{\tau}\delta^{\sigma}_{\alpha}\bar{h}_{\nu\mu;}{}^{\alpha} + 2\delta^{\mu}_{(\rho}\delta^{\nu}_{\tau)}\delta^{\sigma}_{\alpha}\bar{h}^{\alpha}_{\nu;\mu} + \frac{2}{2(n-2)}g_{\rho\tau}g^{\mu\sigma}\bar{h}^{\omega}_{\omega;\mu}\right)_{;\sigma} + 2\Lambda_b\bar{h}_{\rho\tau} \qquad \text{(X.89)}\\
=\; & -\bar{h}_{\tau\rho;}{}^{\alpha}{}_{;\alpha} + 2\bar{h}^{\alpha}_{(\tau;\rho);\alpha} + \frac{1}{(n-2)}g_{\rho\tau}\bar{h}^{\omega}_{\omega;\alpha;}{}^{\alpha} + 2\Lambda_b\bar{h}_{\rho\tau}. \qquad \text{(X.90)}
\end{aligned}
$$

The field equations for $\bar{h}_{\nu\mu}$ can also be obtained by simply substituting the $\mathcal{O}(\bar{h})$ solution (X.62) into the traceless part of the exact field equations (X.3). Using

(R.4,X.62,X.60,X.53,X.54,X.46) gives

$$
\begin{aligned}
0 &= 2\left[H^{\alpha}_{\nu\mu;\alpha} - H^{\alpha}_{\alpha(\nu;\mu)} + \Lambda_b \mathsf{g}_{\nu\mu} - \Lambda_b g_{\nu\mu}\right] &\text{(X.91)}\\
&= 2\left[\frac{1}{2}(\bar{h}_{\alpha\nu;\mu} + \bar{h}_{\mu\alpha;\nu} - \bar{h}_{\nu\mu;\alpha})_{;}{}^{\alpha} + \frac{1}{2(2-n)}(g_{\alpha\nu}\bar{h}^{\tau}_{\tau;\mu} + g_{\mu\alpha}\bar{h}^{\tau}_{\tau;\nu} - g_{\nu\mu}\bar{h}^{\tau}_{\tau;\alpha})_{;}{}^{\alpha}\right.\\
&\quad \left. - \frac{1}{(2-n)}\bar{h}^{\alpha}_{\alpha;(\nu;\mu)} + \Lambda_b h_{\nu\mu}\right] &\text{(X.92)}\\
&= -\bar{h}_{\nu\mu;\alpha;}{}^{\alpha} + 2\bar{h}_{\alpha(\nu;\mu);}{}^{\alpha} + \frac{1}{(n-2)}g_{\nu\mu}\bar{h}^{\tau}_{\tau;\alpha;}{}^{\alpha} + 2\Lambda_b\bar{h}_{\nu\mu}. &\text{(X.93)}
\end{aligned}
$$

Contracting this equation gives

$$\bar{h}^{\tau}_{\tau;\alpha;}{}^{\alpha} = (2-n)(\Lambda_b\bar{h}^{\alpha}_{\alpha} + \bar{h}^{\tau}_{\alpha;\tau;}{}^{\alpha}). \tag{X.94}$$

So we can also write the field equations as

$$0 = -\bar{h}_{\nu\mu;\alpha;}{}^{\alpha} + 2\bar{h}_{\alpha(\nu;\mu);}{}^{\alpha} - g_{\nu\mu}\bar{h}^{\tau}_{\alpha;\tau}{}^{\alpha} + \Lambda_b(2\bar{h}_{\nu\mu} - g_{\nu\mu}\bar{h}^{\alpha}_{\alpha}). \tag{X.95}$$

Now let assume that we can ignore the difference between covariant derivative and ordinary derivative. In that case (X.93,X.95) match the "gauge independent" field equations[66] of linearized gravity except for the Λ_b mass terms. In linearized gravity one often assumes the transverse-traceless gauge condition

$$\bar{h}_{\nu\alpha;}{}^{\alpha} = 0, \quad \bar{h}^{\alpha}_{\alpha} = 0. \tag{X.96}$$

Here we don't have the same freedom because in the $\mathcal{O}(\bar{h})$ coordinate transformation $x^{\nu} \to x^{\nu} + \xi^{\nu}$, the parameter ξ^{ν} has no matrix components, so $\bar{h}_{\nu\mu} \to \bar{h}_{\nu\mu} - \xi_{\nu;\mu} - \xi_{\mu;\nu}$ cannot affect the traceless field $\bar{h}_{\nu\mu}$. However, nothing stops us from seeking solutions which satisfy (X.96), and this assumption also satisfies the contracted field equations (X.94), and it is consistent with the divergence of the field equations (X.95). Assuming (X.96) as a special case, the field equations (X.95) simplify to

$$\bar{h}_{\nu\mu;\alpha;}{}^{\alpha} = 2\Lambda_b\bar{h}_{\nu\mu}. \tag{X.97}$$

Equations (X.97) are like the field equations of massive linearized gravity but with imaginary mass $m = i\hbar\sqrt{2\Lambda_b}$. Below is a z-directed plane-wave solution,

$$\bar{h}_{\nu\mu} = sin(\omega t - kz)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{h}_{+} & \bar{h}_{\times} & 0 \\ 0 & \bar{h}_{\times} & -\bar{h}_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad k^2 - \omega^2 = 2\Lambda_b. \tag{X.98}$$

Since $\Lambda_b > 0$ we have $k > \omega$, and the phase velocity is less than the speed of light,

$$v_{\text{phase}} = \omega/k < 1. \tag{X.99}$$

However, $k > \omega$ is the opposite of what is expected in quantum mechanics where $p = \hbar <k>$, $E = \hbar <\omega>$, $m = \sqrt{E^2 - p^2}$. It also allows the possibility of solutions like

$$\bar{h}_{\nu\mu} = e^{(\alpha t - \kappa z)}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{h}_{+} & \bar{h}_{\times} & 0 \\ 0 & \bar{h}_{\times} & -\bar{h}_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha^2 - \kappa^2 = 2\Lambda_b. \tag{X.100}$$

Because of solutions like this, theories which contain fields with an imaginary mass are commonly labelled as unstable. However, the scalar field ϕ in the Weinberg-Salam theory has an imaginary mass, so this is not necessarily a problem.

Before considering $\bar{h}_{\nu\mu}$ as a Weinberg-Salam ϕ field, let us first assume that waves with imaginary mass are alright. It seems correct that $v_{phase} < 1$ as in (X.99) and it seems reasonable to ignore $e^{\alpha t}$ solutions for the same reason that $e^{\kappa z}$ solutions are ignored for a real mass, i.e. because these solutions satisfy unrealistic boundary conditions. Let us find an effective energy-momentum tensor for $\bar{h}_{\nu\mu}$ and check that the plane-wave solution (X.98) has positive energy. The energy-momentum tensor can be obtained by extracting the $\mathcal{O}(\bar{h}^2)$ components from the exact field equations (X.3). Using (R.4,X.62,X.60,X.76,X.73,X.53,X.54,X.46) gives

$$8\pi\tilde{S}_{\tau\rho} = -tr[K^{\alpha}_{\tau\rho;\alpha} - K^{\alpha}_{\alpha(\tau;\rho)} + H^{\sigma}_{\tau\rho}H^{\alpha}_{\sigma\alpha} - H^{\sigma}_{\tau\alpha}H^{\alpha}_{\sigma\rho} + \Lambda_b\mathsf{g}_{\tau\rho} - \Lambda_b g_{\tau\rho}] \tag{X.101}$$

$$\begin{aligned} &= tr\Bigg[\frac{1}{4(2-n)}\left(g_{\rho\tau}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{;\alpha;}{}^{\alpha} - 2g_{\rho\tau}(\bar{h}^{\omega}_{\omega;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha} + 2(\bar{h}^{\omega}_{\omega;\alpha}\bar{h}_{\rho\tau})_{;}{}^{\alpha}\right) \\ &\quad -\frac{1}{2}\left(((\bar{h}_{\sigma\alpha;\rho} - \bar{h}_{\rho\sigma;\alpha})\bar{h}^{\sigma}_{\tau})_{;}{}^{\alpha} + ((\bar{h}_{\alpha\sigma;\tau} - \bar{h}_{\sigma\tau;\alpha})\bar{h}^{\sigma}_{\rho})_{;}{}^{\alpha} + (\bar{h}_{\rho\tau;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha}\right) \\ &\quad -\left(\frac{1}{2}(\bar{h}^{\sigma}_{\tau;\rho} + \bar{h}^{\sigma}_{\rho;\tau} - \bar{h}_{\tau\rho;}{}^{\sigma}) + \frac{1}{2(2-n)}(\delta^{\sigma}_{\tau}\bar{h}^{\omega}_{\omega;\rho} + \delta^{\sigma}_{\rho}\bar{h}^{\omega}_{\omega;\tau} - g_{\tau\rho}\bar{h}^{\omega}_{\omega;}{}^{\sigma})\right)\frac{\bar{h}^{\alpha}_{\alpha;\sigma}}{(2-n)} \\ &\quad +\left(\frac{1}{2}(\bar{h}^{\sigma}_{\tau;\alpha} + \bar{h}^{\sigma}_{\alpha;\tau} - \bar{h}_{\tau\alpha;}{}^{\sigma}) + \frac{1}{2(2-n)}(\delta^{\sigma}_{\tau}\bar{h}^{\omega}_{\omega;\alpha} + \delta^{\sigma}_{\alpha}\bar{h}^{\omega}_{\omega;\tau} - g_{\tau\alpha}\bar{h}^{\omega}_{\omega;}{}^{\sigma})\right) \\ &\quad \times\left(\frac{1}{2}(\bar{h}^{\alpha}_{\sigma;\rho} + \bar{h}^{\alpha}_{\rho;\sigma} - \bar{h}_{\sigma\rho;}{}^{\alpha}) + \frac{1}{2(2-n)}(\delta^{\alpha}_{\sigma}\bar{h}^{\nu}_{\nu;\rho} + \delta^{\alpha}_{\rho}\bar{h}^{\nu}_{\nu;\sigma} - g_{\sigma\rho}\bar{h}^{\nu}_{\nu;}{}^{\alpha})\right) \\ &\quad -\Lambda_b\left(\bar{h}^{\omega}_{\rho}\bar{h}_{\omega\tau} - g_{\rho\tau}\frac{tr[\bar{h}^{\nu}_{\sigma}\bar{h}^{\sigma}_{\nu}]}{2d(n-2)}\right)\Bigg] \end{aligned} \tag{X.102}$$

$$\begin{aligned} &= tr[g_{\rho\tau}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{;\alpha;}{}^{\alpha}/4(2-n) - g_{\rho\tau}(\bar{h}^{\omega}_{\omega;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha}/2(2-n) \\ &\quad + (\bar{h}^{\omega}_{\omega;\alpha}\bar{h}_{\rho\tau})_{;}{}^{\alpha}/2(2-n) - (\bar{h}_{\sigma\alpha;(\rho}\bar{h}^{\sigma}_{\tau)})_{;}{}^{\alpha} + (\bar{h}_{\rho\sigma}\bar{h}^{\sigma}_{\tau})_{;\alpha;}{}^{\alpha}/2 - (\bar{h}_{\rho\tau;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha}/2 \\ &\quad + \bar{h}^{\sigma}_{\tau;\rho}\bar{h}^{\alpha}_{\alpha;\sigma}[-1/2 + 1/4 + 1/4 + 1/4 - 1/4]/(2-n) \\ &\quad + \bar{h}^{\sigma}_{\rho;\tau}\bar{h}^{\alpha}_{\alpha;\sigma}[-1/2 + 1/4 - 1/4 + 1/4 + 1/4]/(2-n) \\ &\quad + \bar{h}_{\tau\rho;}{}^{\sigma}\bar{h}^{\alpha}_{\alpha;\sigma}[1/2 - 1/4 - 1/4 - 1/4 - 1/4]/(2-n) \\ &\quad + \bar{h}^{\omega}_{\omega;\rho}\bar{h}^{\alpha}_{\alpha;\tau}[(-2-2+1+1+n+1-1-1+1)/4(2-n)^2 + (1+1)/4(2-n)] \\ &\quad + g_{\tau\rho}\bar{h}^{\omega}_{\omega;}{}^{\sigma}\bar{h}^{\alpha}_{\alpha;\sigma}[1/2 - 1/4 - 1/4]/(2-n)^2 \\ &\quad + \bar{h}^{\sigma}_{\tau;\alpha}\bar{h}^{\alpha}_{\sigma;\rho}[1/4 - 1/4] \\ &\quad + \bar{h}^{\sigma}_{\tau;\alpha}\bar{h}^{\alpha}_{\rho;\sigma}[1/4 + 1/4] \\ &\quad + \bar{h}^{\sigma}_{\tau;\alpha}\bar{h}_{\sigma\rho;}{}^{\alpha}[-1/4 - 1/4] \\ &\quad + \bar{h}^{\alpha}_{\tau;\alpha}\bar{h}^{\nu}_{\nu;\rho}[1/4(2-n) - 1/4(2-n)] \end{aligned}$$

$$+\bar h^\sigma_{\alpha;\tau}\bar h^\alpha_{\sigma;\rho}[1/4]$$

$$+\bar h^\sigma_{\alpha;\tau}\bar h^\alpha_{\rho;\sigma}[1/4-1/4]$$

$$+\bar h^\omega_{\omega;\tau}\bar h^\sigma_{\rho;\sigma}[1/4(2{-}n)-1/4(2{-}n)]$$

$$-\Lambda_b(\bar h^\omega_\rho\bar h_{\omega\tau}-g_{\rho\tau}\bar h^\nu_\sigma\bar h^\sigma_\nu/2(n{-}2))], \qquad (X.103)$$

$$\begin{aligned}8\pi\tilde S_{\tau\rho} \;=\;& tr[-g_{\rho\tau}(\bar h^\omega_\nu\bar h^\nu_\omega)_{;\alpha;}{}^\alpha/4(n{-}2)+g_{\rho\tau}(\bar h^\omega_{\omega;\sigma}\bar h^\sigma_\alpha)_;{}^\alpha/2(n{-}2)\\
&-(\bar h^\omega_{\omega;\alpha}\bar h_{\rho\tau})_;{}^\alpha/2(n{-}2)-(\bar h_{\sigma\alpha;(\rho}\bar h^\sigma_{\tau)})_;{}^\alpha+(\bar h_{\rho\sigma}\bar h^\sigma_\tau)_{;\alpha;}{}^\alpha/2-(\bar h_{\rho\tau;\sigma}\bar h^\sigma_\alpha)_;{}^\alpha/2\\
&+\bar h_{\tau\rho;}{}^\sigma\bar h^\alpha_{\alpha;\sigma}/2(n{-}2)-\bar h^\omega_{\omega;\rho}\bar h^\alpha_{\alpha;\tau}/4(n{-}2)+\bar h^\sigma_{\tau;\alpha}\bar h^\alpha_{\rho;\sigma}/2-\bar h^\sigma_{\tau;\alpha}\bar h_{\sigma\rho;}{}^\alpha/2\\
&+\bar h^\sigma_{\alpha;\tau}\bar h^\alpha_{\sigma;\rho}/4-\Lambda_b(\bar h^\omega_\rho\bar h_{\omega\tau}-g_{\rho\tau}\bar h^\nu_\sigma\bar h^\sigma_\nu/2(n{-}2))]. \qquad (X.104)\end{aligned}$$

Contracting this gives

$$\begin{aligned}8\pi\tilde S^\mu_\mu \;=\;& tr[-n(\bar h^\omega_\nu\bar h^\nu_\omega)_{;\alpha;}{}^\alpha/4(n{-}2)+n(\bar h^\omega_{\omega;\sigma}\bar h^\sigma_\alpha)_;{}^\alpha/2(n{-}2)\\
&-(\bar h^\omega_{\omega;\alpha}\bar h^\mu_\mu)_;{}^\alpha/2(n{-}2)-(\bar h^\sigma_{\alpha;\mu}\bar h^\mu_\sigma)_;{}^\alpha+(\bar h^\mu_\sigma\bar h^\sigma_\mu)_{;\alpha;}{}^\alpha/2-(\bar h^\mu_{\mu;\sigma}\bar h^\sigma_\alpha)_;{}^\alpha/2\\
&+\bar h^\mu_{\mu;}{}^\sigma\bar h^\alpha_{\alpha;\sigma}/2(n{-}2)-\bar h^\omega_{\omega;\mu}\bar h^\alpha_{\alpha;}{}^\mu/4(n{-}2)+\bar h^{\sigma\mu}{}_{;\alpha}\bar h^\alpha_{\mu;\sigma}/2-\bar h^\sigma_{\mu;\alpha}\bar h^\mu_{\sigma;}{}^\alpha/2\\
&+\bar h^\sigma_{\alpha;\mu}\bar h^\alpha_{\sigma;}{}^\mu/4-\Lambda_b(\bar h^\omega_\mu\bar h^\mu_\omega-n\bar h^\nu_\sigma\bar h^\sigma_\nu/2(n{-}2))] \qquad (X.105)\\
=\;& tr[(\bar h^\omega_\nu\bar h^\nu_\omega)_{;\alpha;}{}^\alpha(n{-}4)/4(n{-}2)+(\bar h^\omega_{\omega;\sigma}\bar h^\sigma_\alpha)_;{}^\alpha/(n{-}2)\\
&-(\bar h^\omega_\omega\bar h^\mu_\mu)_{;\alpha;}{}^\alpha/4(n{-}2)-(\bar h^\sigma_{\alpha;\mu}\bar h^\mu_\sigma)_;{}^\alpha\\
&+\bar h^\mu_{\mu;}{}^\sigma\bar h^\alpha_{\alpha;\sigma}/4(n{-}2)+\bar h^{\sigma\mu}{}_{;\alpha}\bar h^\alpha_{\mu;\sigma}/2-\bar h^\sigma_{\mu;\alpha}\bar h^\mu_{\sigma;}{}^\alpha/4\\
&-\Lambda_b\bar h^\omega_\mu\bar h^\mu_\omega(n{-}4)/2(n{-}2)]. \qquad (X.106)\end{aligned}$$

So the energy-momentum tensor is

$$8\pi\tilde{T}_{\tau\rho} = 8\pi\left(\tilde{S}_{\tau\rho}-\frac{1}{2}g_{\tau\rho}\tilde{S}^{\mu}_{\mu}\right) \tag{X.107}$$

$$\begin{aligned}
&= tr[-g_{\rho\tau}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{;\alpha;}{}^{\alpha}/8+g_{\rho\tau}(\bar{h}^{\omega}_{\omega}\bar{h}^{\mu}_{\mu})_{;\alpha;}{}^{\alpha}/8(n-2)+g_{\rho\tau}(\bar{h}^{\sigma}_{\alpha;\mu}\bar{h}^{\mu}_{\sigma})_{;}{}^{\alpha}/2\\
&\quad -g_{\rho\tau}\bar{h}^{\mu}_{\mu;}{}^{\sigma}\bar{h}^{\alpha}_{\alpha;\sigma}/8(n-2)-g_{\rho\tau}\bar{h}^{\sigma\mu}{}_{;\alpha}\bar{h}^{\alpha}_{\mu;\sigma}/4+g_{\rho\tau}\bar{h}^{\sigma}_{\mu;\alpha}\bar{h}^{\mu}_{\sigma;}{}^{\alpha}/8\\
&\quad -(\bar{h}^{\omega}_{\omega;\alpha}\bar{h}_{\rho\tau})_{;}{}^{\alpha}/2(n-2)-(\bar{h}_{\sigma\alpha;(\rho}\bar{h}^{\sigma}_{\tau)})_{;}{}^{\alpha}+(\bar{h}_{\rho\sigma}\bar{h}^{\sigma}_{\tau})_{;\alpha;}{}^{\alpha}/2-(\bar{h}_{\rho\tau;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha}/2\\
&\quad +\bar{h}_{\tau\rho;}{}^{\sigma}\bar{h}^{\alpha}_{\alpha;\sigma}/2(n-2)-\bar{h}^{\omega}_{\omega;\rho}\bar{h}^{\alpha}_{\alpha;\tau}/4(n-2)+\bar{h}^{\sigma}_{\tau;\alpha}\bar{h}^{\alpha}_{\rho;\sigma}/2-\bar{h}^{\sigma}_{\tau;\alpha}\bar{h}_{\sigma\rho;}{}^{\alpha}/2\\
&\quad +\bar{h}^{\sigma}_{\alpha;\tau}\bar{h}^{\alpha}_{\sigma;\rho}/4-\Lambda_b(\bar{h}^{\omega}_{\rho}\bar{h}_{\omega\tau}-g_{\rho\tau}\bar{h}^{\nu}_{\sigma}\bar{h}^{\sigma}_{\nu}/4)].
\end{aligned} \tag{X.108}$$

From the field equations (X.90) we get

$$0 = tr[(-\bar{h}_{\nu\rho;}{}^{\alpha}{}_{;\alpha}+2\bar{h}^{\alpha}_{(\nu;\rho);\alpha}+g_{\rho\nu}\bar{h}^{\omega}_{\omega;\alpha;}{}^{\alpha}/(n-2)+2\Lambda_b\bar{h}_{\rho\nu})\bar{h}^{\nu}_{\tau}] \tag{X.109}$$

$$= tr[-\bar{h}_{\nu\rho;}{}^{\alpha}{}_{;\alpha}\bar{h}^{\nu}_{\tau}+\bar{h}^{\alpha}_{\nu;\rho;\alpha}\bar{h}^{\nu}_{\tau}+\bar{h}^{\alpha}_{\rho;\nu;\alpha}\bar{h}^{\nu}_{\tau}+\bar{h}^{\omega}_{\omega;\alpha;}{}^{\alpha}\bar{h}_{\rho\tau}/(n-2)+2\Lambda_b\bar{h}_{\rho\nu}\bar{h}^{\nu}_{\tau}] \tag{X.110}$$

$$\begin{aligned}
&= tr[-\bar{h}_{\nu\rho;}{}^{\alpha}\bar{h}^{\nu}_{\tau}+\bar{h}^{\alpha}_{\nu;\rho}\bar{h}^{\nu}_{\tau}+\bar{h}^{\alpha}_{\rho;\nu}\bar{h}^{\nu}_{\tau}+\bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}_{\rho\tau}/(n-2)]_{;\alpha}\\
&- tr[-\bar{h}_{\nu\rho;}{}^{\alpha}\bar{h}^{\nu}_{\tau;\alpha}+\bar{h}^{\alpha}_{\nu;\rho}\bar{h}^{\nu}_{\tau;\alpha}+\bar{h}^{\alpha}_{\rho;\nu}\bar{h}^{\nu}_{\tau;\alpha}+\bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}_{\rho\tau;\alpha}/(n-2)-2\Lambda_b\bar{h}_{\rho\nu}\bar{h}^{\nu}_{\tau}].
\end{aligned} \tag{X.111}$$

The symmetrization and contraction of this are

$$\begin{aligned}
0 &= tr[-(\bar{h}_{\nu\rho}\bar{h}^{\nu}_{\tau})_{;}{}^{\alpha}/2+\bar{h}^{\alpha}_{\nu;(\rho}\bar{h}^{\nu}_{\tau)}+\bar{h}^{\nu}_{(\tau}\bar{h}^{\alpha}_{\rho);\nu}+\bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}_{\rho\tau}/(n-2)]_{;\alpha}\\
&- tr[-\bar{h}_{\nu\rho;}{}^{\alpha}\bar{h}^{\nu}_{\tau;\alpha}+\bar{h}^{\alpha}_{\nu;(\rho}\bar{h}^{\nu}_{\tau);\alpha}+\bar{h}^{\alpha}_{\rho;\nu}\bar{h}^{\nu}_{\tau;\alpha}+\bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}_{\rho\tau;\alpha}/(n-2)-2\Lambda_b\bar{h}_{\rho\nu}\bar{h}^{\nu}_{\tau}],
\end{aligned} \tag{X.112}$$

$$\begin{aligned}
0 &= tr[-(\bar{h}^{\sigma}_{\nu}\bar{h}^{\nu}_{\sigma})_{;}{}^{\alpha}/2+2\bar{h}^{\alpha}_{\nu;\sigma}\bar{h}^{\nu\sigma}+(\bar{h}^{\omega}_{\omega}\bar{h}^{\sigma}_{\sigma})_{;}{}^{\alpha}/2(n-2)]_{;\alpha}\\
&- tr[-\bar{h}^{\nu}_{\sigma;}{}^{\alpha}\bar{h}^{\sigma}_{\nu;\alpha}+2\bar{h}^{\alpha}_{\nu;\sigma}\bar{h}^{\nu\sigma}{}_{;\alpha}+\bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}^{\sigma}_{\sigma;\alpha}/(n-2)-2\Lambda_b\bar{h}^{\sigma}_{\nu}\bar{h}^{\nu}_{\sigma}].
\end{aligned} \tag{X.113}$$

Adding to (X.108) the expression (X.112)/2$-g_{\rho\tau}$(X.113)/8 gives a simpler form of the energy-momentum tensor which is valid when the effect of source terms in the field equations can be ignored

$$
\begin{aligned}
8\pi\tilde{T}_{\tau\rho} &= tr[-g_{\rho\tau}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{;\alpha;}{}^{\alpha}/8 + g_{\rho\tau}(\bar{h}^{\omega}_{\omega}\bar{h}^{\mu}_{\mu})_{;\alpha;}{}^{\alpha}/8(n-2) + g_{\rho\tau}(\bar{h}^{\sigma}_{\alpha;\mu}\bar{h}^{\mu}_{\sigma})_{;}{}^{\alpha}/2 \\
&\quad - g_{\rho\tau}\bar{h}^{\mu}_{\mu;}{}^{\sigma}\bar{h}^{\alpha}_{\alpha;\sigma}/8(n-2) - g_{\rho\tau}\bar{h}^{\sigma\mu}{}_{;\alpha}\bar{h}^{\alpha}_{\mu;\sigma}/4 + g_{\rho\tau}\bar{h}^{\sigma}_{\mu;\alpha}\bar{h}^{\mu}_{\sigma;}{}^{\alpha}/8 \\
&\quad - (\bar{h}^{\omega}_{\omega;\alpha}\bar{h}_{\rho\tau})_{;}{}^{\alpha}/2(n-2) - (\bar{h}_{\sigma\alpha;(\rho}\bar{h}^{\sigma}_{\tau)})_{;}{}^{\alpha} + (\bar{h}_{\rho\sigma}\bar{h}^{\sigma}_{\tau})_{;\alpha;}{}^{\alpha}/2 - (\bar{h}_{\rho\tau;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha}/2 \\
&\quad + \bar{h}_{\tau\rho;}{}^{\sigma}\bar{h}^{\alpha}_{\alpha;\sigma}/2(n-2) - \bar{h}^{\omega}_{\omega;\rho}\bar{h}^{\alpha}_{\alpha;\tau}/4(n-2) + \bar{h}^{\sigma}_{\tau;\alpha}\bar{h}^{\alpha}_{\rho;\sigma}/2 - \bar{h}^{\sigma}_{\tau;\alpha}\bar{h}_{\sigma\rho;}{}^{\alpha}/2 \\
&\quad + \bar{h}^{\sigma}_{\alpha;\tau}\bar{h}^{\alpha}_{\sigma;\rho}/4 - \Lambda_b(\bar{h}^{\omega}_{\rho}\bar{h}_{\omega\tau} - g_{\rho\tau}\bar{h}^{\nu}_{\sigma}\bar{h}^{\sigma}_{\nu}/4)] \\
&+ tr[-(\bar{h}_{\nu\rho}\bar{h}^{\nu}_{\tau})_{;}{}^{\alpha}/2 + \bar{h}^{\alpha}_{\nu;(\rho}\bar{h}^{\nu}_{\tau)} + \bar{h}^{\nu}_{(\tau}\bar{h}^{\alpha}_{\rho);\nu} + \bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}_{\rho\tau}/(n-2)]_{;\alpha}/2 \\
&- tr[-\bar{h}_{\nu\rho;}{}^{\alpha}\bar{h}^{\nu}_{\tau;\alpha} + \bar{h}^{\alpha}_{\nu;(\rho}\bar{h}^{\nu}_{\tau);\alpha} + \bar{h}^{\alpha}_{\rho;\nu}\bar{h}^{\nu}_{\tau;\alpha} + \bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}_{\rho\tau;\alpha}/(n-2) - 2\Lambda_b\bar{h}_{\rho\nu}\bar{h}^{\nu}_{\tau}]/2 \\
&- g_{\rho\tau}tr[-(\bar{h}^{\sigma}_{\nu}\bar{h}^{\nu}_{\sigma})_{;}{}^{\alpha}/2 + 2\bar{h}^{\alpha}_{\nu;\sigma}\bar{h}^{\nu\sigma} + (\bar{h}^{\omega}_{\omega}\bar{h}^{\sigma}_{\sigma})_{;}{}^{\alpha}/2(n-2)]_{;\alpha}/8 \\
&+ g_{\rho\tau}tr[-\bar{h}^{\nu}_{\sigma;}{}^{\alpha}\bar{h}^{\sigma}_{\nu;\alpha} + 2\bar{h}^{\alpha}_{\nu;\sigma}\bar{h}^{\nu\sigma}{}_{;\alpha} + \bar{h}^{\omega}_{\omega;}{}^{\alpha}\bar{h}^{\sigma}_{\sigma;\alpha}/(n-2) - 2\Lambda_b\bar{h}^{\sigma}_{\nu}\bar{h}^{\nu}_{\sigma}]/8 \qquad \text{(X.114)} \\
&= tr[-g_{\rho\tau}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{;\alpha;}{}^{\alpha}/16 + g_{\rho\tau}(\bar{h}^{\alpha}_{\nu;\sigma}\bar{h}^{\nu\sigma})_{;\alpha}/4 + g_{\rho\tau}(\bar{h}^{\omega}_{\omega}\bar{h}^{\sigma}_{\sigma})_{;}{}^{\alpha}{}_{;\alpha}/16(n-2) \\
&\quad - (\bar{h}_{\sigma\alpha;(\rho}\bar{h}^{\sigma}_{\tau)})_{;}{}^{\alpha}/2 + (\bar{h}_{\rho\sigma}\bar{h}^{\sigma}_{\tau})_{;\alpha;}{}^{\alpha}/4 - (\bar{h}_{\rho\tau;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha}/2 + (\bar{h}^{\nu}_{(\tau}\bar{h}^{\alpha}_{\rho);\nu})_{;}{}^{\alpha}/2 \\
&\quad + \bar{h}^{\sigma}_{\alpha;\tau}\bar{h}^{\alpha}_{\sigma;\rho}/4 - \bar{h}^{\omega}_{\omega;\rho}\bar{h}^{\alpha}_{\alpha;\tau}/4(n-2) - \bar{h}^{\alpha}_{\nu;(\rho}\bar{h}^{\nu}_{\tau);\alpha}/2] \qquad \text{(X.115)}
\end{aligned}
$$

$$
\begin{aligned}
8\pi\tilde{T}_{\tau\rho} &= tr[-g_{\rho\tau}(\bar{h}^{\omega}_{\nu}\bar{h}^{\nu}_{\omega})_{;\alpha;}{}^{\alpha}/16 + g_{\rho\tau}(\bar{h}^{\alpha}_{\nu;\sigma}\bar{h}^{\nu\sigma})_{;\alpha}/4 + g_{\rho\tau}(\bar{h}^{\omega}_{\omega}\bar{h}^{\sigma}_{\sigma})_{;}{}^{\alpha}{}_{;\alpha}/16(n-2) \\
&\quad - (\bar{h}_{\sigma\alpha;(\rho}\bar{h}^{\sigma}_{\tau)})_{;}{}^{\alpha} + (\bar{h}_{\rho\sigma}\bar{h}^{\sigma}_{\tau})_{;\alpha;}{}^{\alpha}/4 - (\bar{h}_{\rho\tau;\sigma}\bar{h}^{\sigma}_{\alpha})_{;}{}^{\alpha}/2 + (\bar{h}^{\nu}_{(\tau}\bar{h}^{\alpha}_{\rho);\nu})_{;}{}^{\alpha}/2 \\
&\quad + \bar{h}^{\sigma}_{\alpha;\tau}\bar{h}^{\alpha}_{\sigma;\rho}/4 - \bar{h}^{\omega}_{\omega;\rho}\bar{h}^{\alpha}_{\alpha;\tau}/4(n-2) + \bar{h}^{\alpha}_{\nu;(\rho|;\alpha}\bar{h}^{\nu}_{|\tau)}/2]. \qquad \text{(X.116)}
\end{aligned}
$$

When averaged over space or time, covariant derivatives commute and gradient terms do not contribute[66], so the averaged energy-momentum tensor is

$$
8\pi<\tilde{T}_{\tau\rho}> \;=\; tr[\,\bar{h}^{\sigma}_{\alpha;\tau}\bar{h}^{\alpha}_{\sigma;\rho}/4 - \bar{h}^{\omega}_{\omega;\rho}\bar{h}^{\alpha}_{\alpha;\tau}/4(n-2) + \bar{h}^{\alpha}_{\nu;\alpha;(\rho}\bar{h}^{\nu}_{\tau)}/2]. \qquad \text{(X.117)}
$$

This result simplifies when $\bar{h}^{\alpha}_{\alpha}=0$ and $\bar{h}^{\alpha}_{\nu;\alpha}=0$ as in our solution (X.98). Note that our averaged energy-momentum tensor matches the result for gravitational waves[66].

It is surprising that neither the non-Abelian character of $\bar{h}_{\nu\mu}$ or the mass term in the field equations affects the energy-momentum tensor. As with gravitational waves, we see from (7.5) that our solution (X.98) has positive energy density,

$$8\pi<\tilde{T}_{00}> = tr[\bar{h}^{\sigma}_{\alpha;0}\bar{h}^{\alpha}_{\sigma;0}]/4 \tag{X.118}$$

$$= tr[\bar{h}^{1}_{1;0}\bar{h}^{1}_{1;0}+\bar{h}^{1}_{2;0}\bar{h}^{2}_{1;0}+\bar{h}^{2}_{1;0}\bar{h}^{1}_{2;0}+\bar{h}^{2}_{2;0}\bar{h}^{2}_{2;0}]/4 \tag{X.119}$$

$$= tr[\bar{h}^{2}_{+}+\bar{h}^{2}_{\times}]\omega^2/2>0. \tag{X.120}$$

Now let us consider the $\bar{h}_{\nu\mu}$ field as a possible replacement of the scalar field ϕ in the Weinberg-Salam theory. In Weinberg-Salam theory the ϕ field has imaginary mass just like $\bar{h}_{\nu\mu}$ in (X.97). The Lagrangian for ϕ is $L_\phi=(\partial\phi/\partial x_\nu)(\partial\phi/\partial x^\nu)/2-V(\phi)$ with potential $V(\phi)=-\mu^2\phi^2/2+|\lambda|\phi^4/4$ so the field equations are

$$0=\partial V/\partial\phi+\partial^2\phi/\partial x^\nu\partial x_\nu=-\mu^2\phi+|\lambda|\phi^3+\partial^2\phi/\partial x^\nu\partial x_\nu. \tag{X.121}$$

These are solved by $\phi_0=\sqrt{\mu^2/|\lambda|}$ which minimizes $V(\phi)$. The potential $V(\phi)$ also has an extremum at $\phi=0$, but the imaginary mass means that this is a maximum rather than a minimum. A weak-field approximation like our analysis with $\bar{h}_{\nu\mu}$ would probe small deviations of ϕ away from $\phi=0$, and this is not really the correct approach for an imaginary mass. Instead, the assumption is that ϕ will "condense" to the constant ϕ_0 which minimizes $V(\phi)$, and the effective Higgs field is then the deviation of ϕ away from this constant value. To see if the $\bar{h}_{\nu\mu}$ field behaves like the ϕ field of Weinberg-Salam theory we must look for a non-zero constant $\bar{h}_{\nu\mu}$ which solves the field equations. For present purposes we will assume cartesian coordinates with $g_{\nu\mu}=\eta_{\nu\mu}$ and $\Gamma^\alpha_{\nu\mu}=0$, so that there is no ambiguity about the meaning of a constant $\bar{h}_{\nu\mu}$. We will ignore the traceful part of the Einstein equations temporarily and hope that these give $g_{\nu\mu} \approx \eta_{\nu\mu}$ when they are subsequently solved. Then using (X.3,X.5,X.54) for a constant $\bar{h}_{\nu\mu}$, the traceless part of the Einstein equations to $\mathcal{O}(\bar{h}^3)$ become

$$
\begin{aligned}
0 &= \frac{1}{2}\Upsilon^{\sigma}_{\nu\mu}\Upsilon^{\alpha}_{\sigma\alpha} + \frac{1}{2}\Upsilon^{\alpha}_{\sigma\alpha}\Upsilon^{\sigma}_{\nu\mu} - \Upsilon^{\sigma}_{\nu\alpha}\Upsilon^{\alpha}_{\sigma\mu} - \frac{I}{d}\,tr\left[\frac{1}{2}\Upsilon^{\sigma}_{\nu\mu}\Upsilon^{\alpha}_{\sigma\alpha} + \frac{1}{2}\Upsilon^{\alpha}_{\sigma\alpha}\Upsilon^{\sigma}_{\nu\mu} - \Upsilon^{\sigma}_{\nu\alpha}\Upsilon^{\alpha}_{\sigma\mu}\right] \\
&+ \Lambda_b\left(\mathsf{g}_{\nu\mu} - \frac{I}{d}tr[\mathsf{g}_{\nu\mu}]\right) \qquad \text{(X.122)} \\
&= \frac{1}{2}\Upsilon^{\sigma}_{\nu\mu}\Upsilon^{\alpha}_{\sigma\alpha} + \frac{1}{2}\Upsilon^{\alpha}_{\sigma\alpha}\Upsilon^{\sigma}_{\nu\mu} - \Upsilon^{\sigma}_{\nu\alpha}\Upsilon^{\alpha}_{\sigma\mu} - \frac{I}{d}\,tr\left[\frac{1}{2}\Upsilon^{\sigma}_{\nu\mu}\Upsilon^{\alpha}_{\sigma\alpha} + \frac{1}{2}\Upsilon^{\alpha}_{\sigma\alpha}\Upsilon^{\sigma}_{\nu\mu} - \Upsilon^{\sigma}_{\nu\alpha}\Upsilon^{\alpha}_{\sigma\mu}\right] \\
&+ \Lambda_b\left(\bar{h}_{\nu\mu} + \bar{h}^{\alpha}_{\nu}\bar{h}_{\alpha\mu} + \bar{h}^{\alpha}_{\nu}\bar{h}^{\sigma}_{\alpha}\bar{h}_{\sigma\mu} - \bar{h}_{\nu\mu}\frac{tr[\bar{h}^{\rho}_{\sigma}\bar{h}^{\sigma}_{\rho}]}{2d(n-2)}\right) - \frac{I\Lambda_b}{d}\,tr\left[\bar{h}^{\alpha}_{\nu}\bar{h}_{\alpha\mu} + \bar{h}^{\alpha}_{\nu}\bar{h}^{\sigma}_{\alpha}\bar{h}_{\sigma\mu}\right] \qquad \text{(X.123)}
\end{aligned}
$$

where the connection equations from (X.29) determine $\Upsilon^{\alpha}_{\nu\mu}$ in terms of $\bar{h}_{\nu\mu}$,

$$
\begin{aligned}
0 &= -\Upsilon^{\rho}_{\beta\mu}(\eta^{\mu\tau} - \bar{h}^{\mu\tau}) - (\eta^{\rho\nu} - \bar{h}^{\rho\nu})\Upsilon^{\tau}_{\nu\beta} - \Upsilon^{\tau}_{\beta\mu}(\eta^{\mu\rho} - \bar{h}^{\mu\rho}) - (\eta^{\tau\nu} - \bar{h}^{\tau\nu})\Upsilon^{\rho}_{\nu\beta} \\
&\quad + \Upsilon^{\alpha}_{\beta\alpha}(\eta^{\rho\tau} - \bar{h}^{\rho\tau}) + (\eta^{\rho\tau} - \bar{h}^{\rho\tau})\Upsilon^{\alpha}_{\beta\alpha}. \qquad \text{(X.124)}
\end{aligned}
$$

There are a couple interesting special cases where we can do an exact calculation of the traceless part of $\Lambda_b\mathsf{g}_{\nu\mu}$ on the second line of (X.122). If $\bar{h}_{\nu\mu}$ has the form of a linearly polarized plane-wave solution (X.98), then using $\tau_i^2 = I$ and $\tau_i\tau_j + \tau_j\tau_i = 0$ from (7.5) we have $\bar{h}^{\alpha}_{\nu}\bar{h}^{\sigma}_{\alpha} = \text{diag}(0, |h_+|^2, |h_+|^2, 0)I$. From the equations used to derive (X.54) we get

$$
\mathsf{g}_{\nu\mu} = (g_{\nu\mu}I + \bar{h}_{\nu\mu})\frac{(1-|h_+|^2)^{1/2}}{(1-|h_+|^2)} = (g_{\nu\mu}I + \bar{h}_{\nu\mu})\frac{1}{(1-|h_+|^2)^{1/2}}\,. \qquad \text{(X.125)}
$$

For an arbitrary traceless Hermitian 2×2 matrix M we have

$$M = \begin{pmatrix} a & b+ic \\ b-ic & -a \end{pmatrix}, \tag{X.126}$$

$$M^2 = (a^2+b^2+c^2)I, \tag{X.127}$$

$$\Delta = \det(1+M) = \det(1-M) = 1-a^2-b^2-c^2, \tag{X.128}$$

$$(I+M)^{-1} = (I-M)/\Delta, \tag{X.129}$$

$$\det[\,\Delta^{1/2}(1+M)^{-1}] = 1. \tag{X.130}$$

If $\bar{h}_{\nu\mu} = g_{\nu\mu}M$ then we have $\bar{h}^\alpha_\nu \bar{h}^\sigma_\alpha = \delta^\sigma_\nu(a^2+b^2+c^2)I$. From (X.45) and the equations above we get

$$\mathsf{g}_{\nu\mu} = g_{\nu\mu}I + \bar{h}_{\nu\mu}, \quad \det(\mathsf{g}_{\nu\mu}) = \Delta^4 \det(g_{\nu\mu}). \tag{X.131}$$

In both cases there is no value of $\bar{h}_{\nu\mu}$ besides $\bar{h}_{\nu\mu}=0$ which zeros out the traceless part of $\Lambda_b\mathsf{g}_{\nu\mu}$ in (X.122). So at least for the special cases considered, the $\bar{h}^3$ terms in (X.123) do not allow a solution to our field equations the way that the ϕ^3 term does with the Weinberg-Salam field equations, and the terms on the first line of (X.123) will have to play an important part.

Let us try to solve the connection equations (X.124). Some simplification gives

$$\begin{aligned} 0 \;=\;& -\Upsilon^\rho_{\beta\mu}\eta^{\mu\tau} - \eta^{\rho\nu}\Upsilon^\tau_{\nu\beta} - \Upsilon^\tau_{\beta\mu}\eta^{\mu\rho} - \eta^{\tau\nu}\Upsilon^\rho_{\nu\beta} + \Upsilon^\alpha_{\beta\alpha}\eta^{\rho\tau} + \eta^{\rho\tau}\Upsilon^\alpha_{\beta\alpha} \\ &+\Upsilon^\rho_{\beta\mu}\bar{h}^{\mu\tau} + \bar{h}^{\rho\nu}\Upsilon^\tau_{\nu\beta} + \Upsilon^\tau_{\beta\mu}\bar{h}^{\mu\rho} + \bar{h}^{\tau\nu}\Upsilon^\rho_{\nu\beta} - \Upsilon^\alpha_{\beta\alpha}\bar{h}^{\rho\tau} - \bar{h}^{\rho\tau}\Upsilon^\alpha_{\beta\alpha}, \end{aligned} \tag{X.132}$$

$$\begin{aligned} 0 \;=\;& -2\Upsilon_{\rho\beta\tau} - 2\Upsilon_{\tau\rho\beta} + 2\Upsilon^\alpha_{\beta\alpha}\eta_{\rho\tau} \\ &+\Upsilon_{\rho\beta\mu}\bar{h}^\mu_\tau + \bar{h}^\nu_\rho\Upsilon_{\tau\nu\beta} + \Upsilon_{\tau\beta\mu}\bar{h}^\mu_\rho + \bar{h}^\nu_\tau\Upsilon_{\rho\nu\beta} - \Upsilon^\alpha_{\beta\alpha}\bar{h}_{\rho\tau} - \bar{h}_{\rho\tau}\Upsilon^\alpha_{\beta\alpha}. \end{aligned} \tag{X.133}$$

Contracting (X.133) over ${}^{\rho}_{\tau}$ gives

$$\Upsilon^{\alpha}_{\beta\alpha} = \frac{1}{2(2-n)}(2\Upsilon^{\sigma}_{\beta\mu}\bar{h}^{\mu}_{\sigma} + 2\bar{h}^{\nu}_{\sigma}\Upsilon^{\sigma}_{\nu\beta} - \Upsilon^{\alpha}_{\beta\alpha}\bar{h}^{\sigma}_{\sigma} - \bar{h}^{\sigma}_{\sigma}\Upsilon^{\alpha}_{\beta\alpha}). \quad \text{(X.134)}$$

Combining (X.133) with it permutations gives

$$\begin{aligned}
0 &= 2\Upsilon_{\rho\beta\tau} + 2\Upsilon_{\tau\rho\beta} - 2\Upsilon^{\alpha}_{\beta\alpha}\eta_{\rho\tau} \\
&- \Upsilon_{\rho\beta\mu}\bar{h}^{\mu}_{\tau} - \bar{h}^{\nu}_{\rho}\Upsilon_{\tau\nu\beta} - \Upsilon_{\tau\beta\mu}\bar{h}^{\mu}_{\rho} - \bar{h}^{\nu}_{\tau}\Upsilon_{\rho\nu\beta} + \Upsilon^{\alpha}_{\beta\alpha}\bar{h}_{\rho\tau} + \bar{h}_{\rho\tau}\Upsilon^{\alpha}_{\beta\alpha} \\
&- 2\Upsilon_{\beta\tau\rho} - 2\Upsilon_{\rho\beta\tau} + 2\Upsilon^{\alpha}_{\tau\alpha}\eta_{\beta\rho} \\
&+ \Upsilon_{\beta\tau\mu}\bar{h}^{\mu}_{\rho} + \bar{h}^{\nu}_{\beta}\Upsilon_{\rho\nu\tau} + \Upsilon_{\rho\tau\mu}\bar{h}^{\mu}_{\beta} + \bar{h}^{\nu}_{\rho}\Upsilon_{\beta\nu\tau} - \Upsilon^{\alpha}_{\tau\alpha}\bar{h}_{\beta\rho} - \bar{h}_{\beta\rho}\Upsilon^{\alpha}_{\tau\alpha} \\
&- 2\Upsilon_{\tau\rho\beta} - 2\Upsilon_{\beta\tau\rho} + 2\Upsilon^{\alpha}_{\rho\alpha}\eta_{\tau\beta} \\
&+ \Upsilon_{\tau\rho\mu}\bar{h}^{\mu}_{\beta} + \bar{h}^{\nu}_{\tau}\Upsilon_{\beta\nu\rho} + \Upsilon_{\beta\rho\mu}\bar{h}^{\mu}_{\tau} + \bar{h}^{\nu}_{\beta}\Upsilon_{\tau\nu\rho} - \Upsilon^{\alpha}_{\rho\alpha}\bar{h}_{\tau\beta} - \bar{h}_{\tau\beta}\Upsilon^{\alpha}_{\rho\alpha}. \quad \text{(X.135)} \\
4\Upsilon_{\beta\tau\rho} &= -2\Upsilon^{\alpha}_{\beta\alpha}\eta_{\rho\tau} - \Upsilon_{\rho\beta\mu}\bar{h}^{\mu}_{\tau} - \bar{h}^{\nu}_{\rho}\Upsilon_{\tau\nu\beta} - \Upsilon_{\tau\beta\mu}\bar{h}^{\mu}_{\rho} - \bar{h}^{\nu}_{\tau}\Upsilon_{\rho\nu\beta} + \Upsilon^{\alpha}_{\beta\alpha}\bar{h}_{\rho\tau} + \bar{h}_{\rho\tau}\Upsilon^{\alpha}_{\beta\alpha} \\
&+ 2\Upsilon^{\alpha}_{\tau\alpha}\eta_{\beta\rho} + \Upsilon_{\beta\tau\mu}\bar{h}^{\mu}_{\rho} + \bar{h}^{\nu}_{\beta}\Upsilon_{\rho\nu\tau} + \Upsilon_{\rho\tau\mu}\bar{h}^{\mu}_{\beta} + \bar{h}^{\nu}_{\rho}\Upsilon_{\beta\nu\tau} - \Upsilon^{\alpha}_{\tau\alpha}\bar{h}_{\beta\rho} - \bar{h}_{\beta\rho}\Upsilon^{\alpha}_{\tau\alpha} \\
&+ 2\Upsilon^{\alpha}_{\rho\alpha}\eta_{\tau\beta} + \Upsilon_{\tau\rho\mu}\bar{h}^{\mu}_{\beta} + \bar{h}^{\nu}_{\tau}\Upsilon_{\beta\nu\rho} + \Upsilon_{\beta\rho\mu}\bar{h}^{\mu}_{\tau} + \bar{h}^{\nu}_{\beta}\Upsilon_{\tau\nu\rho} - \Upsilon^{\alpha}_{\rho\alpha}\bar{h}_{\tau\beta} - \bar{h}_{\tau\beta}\Upsilon^{\alpha}_{\rho\alpha}. \quad \text{(X.136)}
\end{aligned}$$

It is unclear how to use this to obtain an analytical expression for $\Upsilon^{\alpha}_{\nu\mu}$. We will have to leave this work unfinished.

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